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THE SELBERG TRACE FORMULA FOR $SL(3, \mathbb{Z})\backslash SL(3, \mathbb{R})/SO(3, \mathbb{R})$

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ABSTRACT. In this paper we compute the trace formula for $SL(3, \mathbb{Z})$ in detail and refine it to a greater extent than has previously been done. We show that massive cancellation occurs in the parabolic terms, leading to a far simpler formula than had been thought possible.

1. INTRODUCTION AND BACKGROUND

"Of course, we won't really understand the trace formula until it is written down for $SL(4, \mathbb{Z})$." Dennis Hejhal

The Selberg trace formula for groups of rank greater than one has been under attack for some time by several people. Arthur has made the most progress (see [1–4]), but others have also contributed, for example, Osborne and Warner [8], Flicker [7], Warner [34], and Efrat [6]. Of these, Efrat's work is closest in spirit to this paper, in that it sticks to a particular case and aims to get a formula usable to the classical number theorist. The trace formula, as anyone reading this paper is apt to know, computes the trace, on the discrete spectrum of the Laplace-Beltrami operator on a particular space, of a cleverly designed integral operator. The cleverly designed operator is supposed to be, above all else, trace class.

In Arthur's and Flicker's papers this operator is assumed to be trace class, which can be arranged in the adelic case by using a cusp form as the real part of the function which defines the operator. For applications such as Sarnak's [20] and those described in the appendices of Hejhal [9, 10], this restriction will not do. Müller [17] has now solved the trace class conjecture, so in principal any function can now be put into Author's formula (or Flicker's, Osborne and Warner's, etc.) with complete a priori justification.

To do these same applications, however, one needs more than to know the formula converges; one must also determine to what it converges. For such applications one needs a formula as completely explicit as Selberg's original [22]. The formula given in this paper is such a formula.

The main obstacle to deriving a clear trace formula for these groups is the presence of a truncation parameter in all terms in the formula coming from orbital integrals and Eisenstein series. One expects all the nonconvergent terms

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to sum to a convergent whole as this parameter tends to infinity. We believe this paper sheds a great deal of light on exactly how this happens. Calculations such as those in Appendix 1 and §7 are not foreshadowed by previous work and are the key to cancellation. They could well be generalized to other groups now that we see one example in detail. The obstacles to the general case should now be quite visible.

In addition to Flicker and this author, two others have worked on the case of $SL(3, \mathbb{R})$ specifically. One is Kolk [13], who derived a trace formula for co-compact subgroups and the other is Venkov [25], who worked specifically on $SL(3, \mathbb{Z})$ but was, again, unable to find a trace class operator. Nonetheless we are indebted to him for his clear description of the Eisenstein series for $SL(3, \mathbb{Z})$ which we use extensively here. Also we refer any reader who is not acquainted with the classical Selberg trace formula to Selberg [22], Hejhal [9, 10], or Terras [23] for a description of it, because we will be needing it later. And now, onward!

Let G be a real semisimple Lie group. G has a maximal compact subgroup K and G/K is a Riemannian symmetric space. This means that G/K has a G-invariant metric and measure, (G acting on the left), as well as an associated collection of G-invariant differential operators. In fact, the algebra of invariant differential operators on G (and hence on G/K) is equivalent to the center of the enveloping algebra $U(G)$ which in general has rank G generators. So for $SL(3, \mathbb{R})$ there are two generators of $U(G)$, one of which is the Laplace-Beltrami second-order operator and the other is third order. When we speak of the "spectrum" on some space we will in fact mean the joint spectrum of these two operators which we denote by Δ_1 and Δ_2.

A function, f, on the coset space G/K can be interpreted as a function on G which is invariant on the right by K. In addition, we can require f to be invariant by K on the left. Such a function is called K-bi-invariant and from now on f will be such a function.

Let $Y_1, Y_2 \in G/K$. We will write Y_1^{-1} to stand for the inverse of Y_1, where Y_1 is a coset representative in G. Then the object $f(Y_1^{-1}Y_2)$ makes sense for all K-bi-invariant f because it is independent of the choice of coset representatives for Y_1 and Y_2. In this way we can construct a point pair invariant which is the convolution kernel of an integral operator by setting $k(Y_1, Y_2) = f(Y_1^{-1}Y_2)$ which is a point pair invariant and the kernel of

$$L_k(g)(Y_1) = \int_{G/K} k(Y_1, Y_2) g(Y_2)\, d\mu(Y_2).$$

We will assume f is smooth with compact support and we will take $d\mu$ to be a left invariant measure on G/K.

In general, however, we wish to form our operator out of a function which is invariant under the left action of some discrete group Γ. To do this we start with f as before and form $F(Y) = \sum_{\gamma\in\Gamma} f(\gamma Y)$ and get

$$K(Y_1, Y_2) = \sum_{\gamma\in\Gamma} f(Y_1^{-1}\gamma Y_2) \tag{1}$$

and

$$L_k(g)(Y_1) = \int_{\mathfrak{I}} k(Y_1, Y_2) g(Y_2)\, d\mu(Y_2).$$

Note that k is still K-invariant although F is not, and k is left invariant under Γ in both variables. $\mathfrak{J}$ is the fundamental region for the action of Γ on G/K. We think of $g \in L^2(\Gamma\backslash G/K)$.

Let Φ be an eigenfunction of Δ_1 and Δ_2 which is invariant under Γ. Because L_k commutes with these operators, Φ is an eigenfunction of it also. Let λ_1, λ_2, h denote the eigenvalues

$$\Delta_1\Phi = \lambda_1\Phi, \quad \Delta_2\Phi = \lambda_2\Phi, \quad L_k\Phi = h\Phi$$

for the case where $G = SL(3, \mathbb{R})$, $K = SO(3, \mathbb{R})$. It is a theorem of Selberg (called Selberg's Lemma) that h depends only on λ_1 and λ_2 and not on the choice of eigenfunction ϕ corresponding to these eigenvalues. This is true for both types of kernels mentioned so far. The function $h(\lambda_1, \lambda_2)$ is called the *Selberg transform* of f.

We will be making use of several kinds of coordinates on $SL(3, \mathbb{R})/SO(3, \mathbb{R})$ which we describe now. Langlands [16] introduced several kinds of coordinates on $SL(n, \mathbb{R})/SO(n, \mathbb{R})$ that specialize to three kinds for $n = 3$. First note that the Iwasawa decomposition allows us to write $G = A^+NK$ where A is diagonal, N strictly upper triangular, and $K = SO(3, \mathbb{R})$ for this case. Choosing a coset amounts to choosing an element of N and one of A^+, the identity component of A. Thus we can identify G/K in this case with the coordinates

$$Y = \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & 1/y_1y_2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix}$$

where $y_1, y_2 > 0$. Notice that the elements of the form Y can also be viewed as a subgroup of G although "multiplication of cosets" makes no sense. The set $\{Y = an;\ a \in A,\ n \in N\}$ is an example of a parabolic subgroup of G.

In general the parabolic subgroups of G are normalizers of the various closed reductive subalgebras $\widetilde{M}$ of G where $\widetilde{M}$ contains some conjugate A. The standard parabolic subgroups are those where $\widetilde{M}$ actually contains A and every parabolic subgroup is conjugate to a standard one. For $G = SL(3, \mathbb{R})$ the candidates for $\widetilde{M}$ and P are

$$\widetilde{M}_0 = A = \left\{ \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & 1/y_1y_2 \end{pmatrix} \right\}, \qquad P(M_0) = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\},$$

$$\widetilde{M}_1 = \left\{ \left(\begin{array}{cc|c} * & * & 0 \\ * & * & 0 \\ \hline 0 & 0 & * \end{array}\right) \right\}, \qquad P(M_1) = \left\{ \left(\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline 0 & 0 & * \end{array}\right) \in G \right\},$$

$$\widetilde{M}_2 = \left\{ \left(\begin{array}{c|c|c} * & 0 & 0 \\ \hline 0 & * & * \\ 0 & * & * \end{array}\right) \right\}, \qquad P(M_2) = \left\{ \left(\begin{array}{c|c|c} * & * & * \\ \hline 0 & * & * \\ 0 & * & * \end{array}\right) \in G \right\},$$

$$\widetilde{M}_3 = G, \qquad P(M_3) = G.$$

Langlands' coordinates for P_1 and P_2 are based on a convenient decomposition and are given below for P_1. We write

$$P(M_1) = M_1A_1N_1$$

where

$$P(M_1) = \left\{ \left(\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline 0 & 0 & * \end{array} \right) \in G \right\},$$

$$A_1 = \left\{ \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_1 & 0 \\ 0 & 0 & \gamma_1^{-2} \end{pmatrix}, \ \gamma_1 > 0 \right\},$$

$$N_1 = \left\{ \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}, \ x_1, t_1 \in \mathbb{R} \right\},$$

$$M_1 = \left\{ \left(\begin{array}{cc|c} * & * & 0 \\ * & * & 0 \\ \hline 0 & 0 & \pm 1 \end{array} \right), \ \det \begin{pmatrix} * & * \\ * & * \end{pmatrix} = \pm 1 \right\}.$$

Notice that M_1 is a real reductive group with compact center and in this case we have

$$M_1/M_1 \cup K \cong SL(2, \mathbb{R})/SO(2, \mathbb{R})$$

so natural coordinates for $SL(3, \mathbb{R})/SO(3, \mathbb{R})$ are given by

$$(2) \qquad \begin{pmatrix} u_1^{1/2} & vu_1^{-1/2} & 0 \\ 0 & u_1^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 & & \\ & \gamma_1 & \\ & & \gamma_1^{-2} \end{pmatrix} \begin{pmatrix} 1 & 0 & x_1 \\ & 1 & t_1 \\ & & 1 \end{pmatrix}$$

where $v + iu$ can be regarded either as a point in the Poincaré upper half plane or as a coset representative for $M_1/M_1 \cap K$. Similarly, consideration of $P(M_2)$ yields coordinates

$$(3) \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & u_2^{1/2} & v_2 u_2^{-1/2} \\ 0 & 0 & u_2^{-1/2} \end{pmatrix} \begin{pmatrix} \gamma_2^2 & 0 & 0 \\ 0 & \gamma_2^{-1} & 0 \\ 0 & 0 & \gamma_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & t_2 & x_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Associated to the discrete group $SL(3, \mathbb{Z})$ we also have parabolic subgroups. We label the standard ones

$$(4) \qquad \begin{gathered} P_0 = SL(3, \mathbb{Z}) \cap P(M_1), \quad P_1 = SL(3, \mathbb{Z}) \cap P(M_1), \\ P_2 = SL(3, \mathbb{Z}) \cap P(M_2). \end{gathered}$$

Obviously both P_1 and P_2 contain P_0. For further information on parabolic subgroups of a Lie group, see Schlichtkrull [21].

In order to build a trace class operator on $L^2(\Gamma \backslash G/K)$ we must investigate the joint spectral decomposition of Δ_1 and Δ_2 on this space. Because $\Gamma \backslash G/K$ is not a compact manifold in our case, the joint spectrum consists of a discrete spectrum and a continuous spectrum. Furthermore the continuous spectrum has a two dimensional part and an infinite number of one dimensional parts. We now need to describe various orthogonal subspaces of $L^2(\Gamma \backslash G/K)$. Let $\Psi(\gamma_1(Y), u_1(Y))$ be a function in $C_c^\infty(A)$. Notice that P_0 fixes γ_1 and u_1 both. So to build an automorphic function out of Ψ we must sum over $P_0 \backslash \Gamma$, getting

$$(5) \qquad \theta_\Psi(Y) = \sum_{\gamma \in P_0 \backslash \Gamma} \Psi(\gamma_1(\gamma y), u_1(\gamma y)).$$

Let Θ_0 be the closed subspace spanned by all θ_Ψ. These Ψ are precisely the functions which, if averaged over K, would yield the K-bi-invariant functions on G.

Next, let Ψ be a C_c^∞ function of y_1 alone and let $\nu(z_1)$ be any even cusp form for $SL(2, \mathbb{Z})$ acting on the variable $z_1 = v_1 + iu_1$. Then Ψ and ν are both invariant under the action of P_1 so let

$$\theta_{1,\Psi,\nu}(Y) = \sum_{\gamma \in P_1\backslash\Gamma} \Psi(y_1(\gamma Y)) \cdot \nu(z_1(\gamma Y)) \tag{6}$$

and, similarly

$$\theta_{2,\Psi,\nu}(Y) = \sum_{\gamma \in P_2\backslash\Gamma} \Psi(y_2(\gamma Y))\nu(z_2(\gamma Y))$$

where $z_2 = v_2 + iu_2$. Recall that a cusp form for $SL(2, \mathbb{Z})$ is an eigenfunction of the Laplace-Beltrami operator on $SL(2, \mathbb{Z})\backslash SL(2, \mathbb{R})/SO(2, \mathbb{R})$ which dies out as $\operatorname{Im} z \to \infty$. Equivalently, when expanded as a Fourier series in $x = \operatorname{Re}(z)$ the constant term is zero. Again, equivalently, the integral of ν along any closed N-orbit (closed under the action of Γ) is zero. Let $\Theta_{1,2}$ be the closed subspace spanned by functions of the form $\theta_{1,\Psi,\nu}$ and $\theta_{2,\Psi,\nu}$. Ultimately we will see that the trace formula requires only one of these, because of the functional equations relating them. Let $y_1 = \gamma_1^6$, $y_2 = \gamma_2^6$. Then Haar measure on G/K is given in Langlands' coordinates by

$$d\mu(Y) = \frac{dy_1\, dz_1\, dx_1\, dt_1}{y_1^2 u_1^2} = \frac{dy_2\, dz_2\, dx_2\, dt_2}{y_2^2 u_2^2}. \tag{7}$$

Where $z_i = v_i + iu_i$. Let $\mathfrak{J}$ be a fundamental region for Γ. Then Θ_0 and $\Theta_{1,2}$ are orthogonal with respect to the inner product on $L^2(\Gamma\backslash G/K)$ given by

$$\langle f, g \rangle = \int_{\mathfrak{J}} f(Y) g(Y)\, d\mu(Y). \tag{8}$$

So we now have

$$L^2(\Gamma\backslash G/K) = \mathbb{H} = \mathbb{H}_0 \oplus \Theta_0 \oplus \Theta_{1,2}.$$

and we would like to characterize all $f \in \mathbb{H}_{0_0} = (\Theta_0 \oplus \Theta_{1,2})^\perp$.

Let $f(Y) \in \mathbb{H}_0$. We have

$$\begin{aligned} 0 &= \langle \theta_{\Psi_1}(Y), f(Y) \rangle \\ &= \int_{\mathfrak{J}} \left(\sum_{\gamma \in P_0\backslash\Gamma} \Psi_1(y_1(\gamma Y), u_1(\gamma Y)) \right) \overline{f(Y)}\, d\mu(Y) \\ &= \int_{\mathfrak{J}_0} \Psi_1(y_1(Y), u_1(Y)) \overline{f(Y)}\, d\mu(Y) \\ &= \int_{y_1=0}^{\infty} \int_{u_1=0}^{\infty} \Psi_1(y_1(Y), u_1(Y)) \int_{x_1=}^{1/2} \int_{-1/2=}^{1/2} \int_{t_1=v_1}^{1/2} \overline{f(Y)}\, dx_1\, dt_1\, dv_1 \frac{dy_1\, du_1}{y_1^2 u_1^2} \end{aligned}$$

which is true for all $\Psi_1 \in C_c^\infty(A)$ if and only if

$$\int_{x_1=-1/2}^{1/2} \int_{t_1=-1/2}^{1/2} \int_{v_1=-1/2}^{1/2} f(Y)\, dx_1\, dt_1\, dv_1 = 0.$$

Similarly, we have

$$\begin{aligned} 0 &= \langle \theta_{1,\psi_2,v}, f(Y)\rangle \\ &= \int_{\mathfrak{I}} \left(\sum_{\gamma\in P_1\backslash\Gamma} \Psi_2(y_1(\gamma Y))v(z, (\gamma Y))\right) f(Y)\, d_\mu(Y) \\ &= \int_{\mathfrak{I}_1} \Psi_2(y_1(Y))v(z, (Y))f(Y)\, d_\mu(Y) \end{aligned}$$

where $\mathfrak{I}_i$ is a fundamental region for the action of P_i. Four copies of $\mathfrak{I}_1$ fit into the region $\{y_1 > 0, -1/2 \le x_1 < 1/2, -1/2 \le t_1 < 1/2, v_1 + iu_1 \in \widetilde{\mathfrak{I}}\}$ with possible overlaps on the boundaries and with $\widetilde{\mathfrak{I}}$ the fundamental region for $SL(2, \mathbb{Z})$ in the Poincaré upper half-plane. So we have

$$\begin{aligned} 0 &= \frac{1}{4}\int_{y_1=0}^{\infty} \iint_{z\in\widetilde{\mathfrak{I}}} \int_{x_1=-1/2}^{1/2} \int_{t_1=-1/2}^{1/2} \Psi_2(y_1(Y))v(z_1(Y))\overline{f(Y)}\, dt_1\, dx_1 \frac{du_1\, dv_1}{u_1^2}\frac{dy_1}{y_1^2} \\ &= \frac{1}{4}\int_{y_1=0}^{\infty} \Psi_1(y_1(Y)) \int_{z_1\in\widetilde{\mathfrak{I}}} v(z_1(Y)) \left(\int_{x_1=-1/2}^{1/2} \int_{t_1=-1/2}^{1/2} \overline{f(Y)}\, dt_1\, dx_1 \right) \frac{du_1\, dv_1}{u_1^2}\frac{dy_1}{y_1^2}. \end{aligned}$$

Since we can choose any Ψ_1 it follows that

$$0 = \int_{z_1\in\widetilde{\mathfrak{I}}} v(z_1(Y)) \left(\int_{x_1=-1/2}^{1/2} \int_{t_1=-1/2}^{1/2} \overline{f(Y)}\, dx_1\, dt_1 \right) \frac{du_1\, dv_1}{u_1^2}$$

for all y_1. Fixing y_1, the function

$$g(z_1) = \int_{x_1=-1/2}^{1/2} \int_{t_1=-1/2}^{1/2} \overline{f(Y)}\, dx_1\, dt_1$$

is a function of z_1 alone. Thus it is in the span of the even cusp forms and Eisenstein series and constants on $SL(2, \mathbb{Z})\backslash SL(2, \mathbb{R})/SO(2, \mathbb{R})$. (Notice f is even.) It cannot be in the span of the even cusp forms, otherwise we can pick v to give a nonzero inner product. If it is in the span of the constants or Eisenstein series then it cannot be orthogonal to Θ_0. Therefore we must have that

$$\int_{x_1=-1/2}^{1/2} \int_{t_1=-1/2}^{1/2} f(Y)\, dx_1\, dt_1 = 0.$$

A similar argument for $\theta_{2,\psi_2,v}$ shows that

$$\int_{x_2=-1/2}^{1/2} \int_{t_2=-1/2}^{1/2} f(Y)\, dx_2\, dt_2 = 0.$$

So we have shown that $f \in \mathbb{H}_0$ if and only if

$$\begin{aligned} 0 &= \int_{x_1=-1/2}^{1/2} \int_{t_1=-1/2}^{1/2} \int_{v_1=-1/2}^{1/2} f(Y)\, dx_1\, dt_1\, dv_1 \\ &= \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} f(Y)\, dx_1\, dt_1 = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} f(Y)\, dx_2\, dt_2. \end{aligned} \tag{9}$$

These equalities tell us that the constant term in the Fourier expansion of $f \in \mathbb{H}_0$ is zero whether we expand with respect to x_1 and t_1, x_2 and t_2, or x_1, t_1, and v_1. If there are eigenfunctions in $\mathbb{H}_0$, they will be called cuspidal, or cusp forms, with respect to the group $SL(3, \mathbb{Z})$.

A concise discussion of this decomposition can be found in Venkov [25].

We now construct the Eisenstein series associated to θ_0. Set

$$E_0(Y; s, t) = \sum_{\gamma \in P_0 \backslash \Gamma} y_1(\gamma Y)^s u_1(\gamma Y)^t. \tag{10}$$

Let $L_\Psi(s, t)$ be the Mellin transform $\psi(y_1, u_1)$, that is,

$$L_\Psi(s, t) = \int_0^\infty \int_0^\infty \Psi(y_1, u_1) y_1^{-s-1} u_1^{-t-1}\, dy_1\, du_1.$$

Then

$$\Psi(y_1, u_1) = \frac{-1}{4\pi^2} \int_{\mathrm{Re}\, s=s_0} \int_{\mathrm{Re}\, t=t_0} L_\Psi(x, t) y_1^s u_1^t\, ds\, dt.$$

Therefore

$$\begin{aligned}
\theta_\Psi(Y) &= \sum_{P_0 \backslash \Gamma} \Psi(y_1(\gamma Y), u_1(\gamma Y)) \\
&= \frac{-1}{4\pi^2} \sum_{P_0 \backslash \Gamma} \int_{\mathrm{Re}\, s=s_0} \int_{\mathrm{Re}\, t=t_0} L_\Psi(s, t) y_1(\gamma Y)^s u_1(\gamma Y)^t\, ds\, dt \\
&= \frac{-1}{4\pi^2} \int_{\mathrm{Re}\, s=s_0} \int_{\mathrm{Re}\, t=t_0} L_\Psi(s, t) E_0(Y; s, t)\, ds\, dt
\end{aligned}$$

providing the sum which defines $E_0(Y; s, t)$ converges. As described in [25], the Eisenstein series is known to converge for $3s - t > 2$, $t > 1$, but can be meromorphically continued. These series have infinitely many polar lines but the three which will cause us trouble later are given by the equations

$$t = 1, \qquad 3s - t = 2, \qquad 3s + t = 3. \tag{11}$$

In the rest of this paper we will be computing a lot of line integrals involving Eisenstein series, along paths of the sort $\mathrm{Re}\, s = c_1$. To do these computations it will be useful to move a line like this to $\mathrm{Re}\, s = \frac{1}{2}$. In doing so we pick up residues of the Eisenstein series in (for example) the variable s at the various poles. The details to the functions obtained as residues in this way are given in §5, but for now we keep the discussion brief. These residues are Eisenstein series of one variable in general. They are orthogonal to the span of the part of the continuous spectrum given by

$$\mathrm{span}\left\{E_0\left(Y; \frac{1}{2} + ir_1, \frac{1}{2} + ir_2\right)\right\} = \mathbf{\Theta}_0^{(2)}$$

and thus we have the decomposition $\mathbf{\Theta}_0 = \mathbf{\Theta}_0^{(1)} \oplus \mathbf{\Theta}_0^{(2)}$ where

$$\mathbf{\Theta}_0^{(1)} = \mathrm{span}\{\text{residues of } E_0\}.$$

Next, let v be an even cups form for $SL(2, \mathbb{Z})$ on the upper half-plane. We let

$$E_1(Y; \nu, s) = \sum_{P_1 \backslash \Gamma} y_1(\gamma Y)^s \nu(z_1(\gamma Y)) \tag{12}$$

and

$$E_2(Y;\nu,s)=\sum_{P_1\backslash\Gamma} y_2(\gamma Y)^s\nu(z_2(\gamma Y)).$$

These series converge for $\operatorname{Re}s>1$ and have any poles which are in $\operatorname{Re}s\geq\frac{1}{2}$ right on the real axis in the interval $(\frac{1}{2},1]$. The poles do not depend on the choice of ν in a given eigenspace. E_1 and E_2 are related by the functional equations

$$\begin{aligned}E_1(Y;c_2(\nu,s),1-s)&=E_2(Y;\nu,s),\\E_2(Y;c_1(\nu,s),1-s)&=E_1(Y;\nu,s),\end{aligned}\tag{13}$$

where $c_j(\nu,s)$ is some choice of eigenfunction in the same eigenspace as ν and

$$c_1(c_2(\nu;s);1-s)=c_2(c_1(\nu;s);1-s)=\nu.$$

All descriptions of Eisenstein series are explained in Langlands [16]. Due to these functional equations we can span the set of all functions of the form

$$\theta_{i,\Psi,\nu_\lambda}=\sum_{p_i\backslash\Gamma}\Psi(y_i(Y))\nu_\lambda(z_i(Y)),$$

where ν_λ belongs to the eigenspace with given eigenvalue λ, by choosing an orthonormal basis for the ν_λ and fixing $i=1$. Furthermore, if we set

$$L_\Psi(s)=\int_0^\infty\Psi(y)y^{-s-1}\,dy,$$

then

$$\Psi(y)=\frac{1}{2\pi i}\int_{\operatorname{Re}s=s_0}L_\Psi(s)y^s\,ds$$

and hence

$$\begin{aligned}\theta_{1,\Psi,\nu_\lambda}(Y)&=\sum_{P_1\backslash\Gamma}\Psi(y,(\gamma Y))\nu(z_1(\gamma Y))\\&=\sum_{P_1\backslash\Gamma}\frac{1}{2\pi i}\left[\int_{\operatorname{Re}s=s_0}L_\Psi(s)y_1(\gamma Y)^s\,ds\right]\nu(z_i(\gamma Y))\\&=\frac{1}{2\pi i}\int_{\operatorname{Re}s=s_0}L_\Psi(s)E_1(Y;\nu,s)\,ds\end{aligned}$$

and therefore $\Theta_{1,2}$ is spanned by the E_1 if $\operatorname{Re}s>1$ and we choose an orthogonal basis $\{\nu_k\}$ for each eigenspace.

At some point later in this paper we will wish to move the contour of integration to $\operatorname{Re}s=\frac{1}{2}$ but in doing so we pick up residues which occur discreetly. Hence $\Theta_{1,2}$ breaks up into two orthogonal pieces,

$$\Theta_{1,2}=\Theta_{1,2}^{(1)}\oplus\Theta_{1,2}^{(2)}\tag{14}$$

where $\Theta_{1,2}^{(1)}$ is spanned by residues of Eisenstein series in $(\frac{1}{2},1]$ and $\Theta_{1,2}^{(2)}$ is spanned by the E_1 along $\operatorname{Re}s=\frac{1}{2}$.

Thus we have the decomposition

$$H=H_0\oplus\Theta_{1,2}^{(1)}\oplus\Theta_{1,2}^{(2)}\oplus\Theta_0^{(1)}\oplus\Theta_0^{(2)}.\tag{15}$$

All the Θ are spanned by various joint eigenfunctions of the invariant differential operators and consequently each subspace described above is preserved under the action of our invariant integral operator. Thus if H_0 is nonempty we conclude it contains some eigenfunctions. It is a theorem of Gelfand and Piatetski-Shapiro that $L_k P_0$, where P_0 is projection onto H_0, is an operator of Hilbert-Schmidt type and hence has discrete spectrum. (See [8].)

At this point we should note that the referee of this paper sketched a proof that, in fact, there are no poles of E_1 or E_2 when $\operatorname{Re} s > \frac{1}{2}$ and that therefore $\Theta^{(1)}_{1,2}$ is empty.

This will not affect our calculation in any way, but we left the $\Theta^{(1)}_{1,2}$ in the above formula to emphasize the fact that the trace formula we are about to derive does not depend on projecting our operator onto the cuspidal part of the spectrum. Nonetheless we are much indebted to the patient referee who took the time to point this out.

We point out that the discrete spectrum is contained in $H_0 \cup \Theta^{(1)}_{1,2}$ with the possible exception of the constant function which occurs in $\Theta^{(1)}_0$. Also we note that the space $\Theta^{(2)}_{1,2}$ decomposes under an orthogonal decomposition of the ν so that we have

$$\Theta^{(2)}_{1,2} = \sum_{K=1}^{\infty} \Theta_{1,2}(\nu_K)$$

where $\Theta_{1,2}(\nu_K)$ is just the span of those elements of $\Theta^{(2)}_{1,2}$ with some fixed cusp form ν_K. The proofs of this and also the orthogonality of Θ_0 and $\Theta_{1,2}$ follow from the Fourier expansions of the various Eisenstein series, which will be presented later.

It follows from the above discussion that if we wish to adjust our integral operator L_k we must subtract off the contribution from the continuous spectrum. To see what this contribution looks like, we will compute the action of L_k on an element of $\Theta^{(2)}_0$. Recall

$$\begin{aligned}
L_k(\theta_0, \Psi)(Y_1) &= \int_{\mathfrak{I}} k(Y_1, Y_2)\theta_{0,\Psi}(Y_2)\, d_\mu(Y_2) \\
&= \int_{\mathfrak{I}} \left(\sum_\gamma f(Y_1^{-1}\gamma Y_2)\right) \theta_{0,\Psi}(Y_2)\, d_\mu(Y_2) \\
&= \int_{\mathfrak{I}} \left(\sum_\gamma f(Y_1^{-1}\gamma Y_2)\right) \int_{\operatorname{Re} s=1/2}\int_{\operatorname{Re} t=1/2} L_\Psi(s, t) E(Y_2; s, t)\, ds\, dt\, d_\mu(Y_2) \\
&= \int_{\operatorname{Re} s=1/2}\int_{\operatorname{Re} t=1/2} L_\Psi(s, t) \left(\int_{\mathfrak{I}} k(Y_1, Y_2) E(Y_2; s, t)\, d_\mu(Y_2)\right) ds\, dt \\
&= \int_{\operatorname{Re} s=1/2}\int_{\operatorname{Re} t=1/2} L_\Psi(s, t) h(s, t) E(Y_1; s, t)\, ds\, dt \\
&= \int_{\mathfrak{I}}\int_{\operatorname{Re} s=1/2}\int_{\operatorname{Re} t=1/2} \theta_{0,\Psi}(Y_2) E(Y_2; 1-s, 1-t) \\
&\qquad\qquad \times h(s, t) E(Y_1; s, t)\, ds\, dt\, d1/2.
\end{aligned}$$

Thus if we wish to subtract off the contribution from $\Theta_0^{(2)}$ we must use the operator constructed from the kernel

$$\begin{aligned} &k(Y_1, Y_2) - \int_{\mathrm{Re}\, s=1/2} \int_{\mathrm{Re}\, t=1/2} h(s, t) E(Y_2; 1-s, 1-t) E(Y_1; s, t)\, ds\, dt \\ &\quad = k(Y_1, Y_2) - \int_{\mathrm{Re}\, s=1/2} \int_{\mathrm{Re}\, t=1/2} h(s, t) E(Y_1; s, t) \overline{E(Y_2; s, t)}\, ds\, dt. \end{aligned}$$

The calculation is similar for $\Theta_{1,2}^{(2)}$ and $\Theta_0^{(1)}$ although we have yet to characterize the elements of $\Theta_0^{(1)}$.

So the operator whose trace we will compute is the one formed from the kernel

(16)

$$\begin{aligned} \tilde{k}(Y_1, Y_2) = k(Y_1, Y_2) &- \frac{1}{4\pi} \int_{\mathrm{Re}\, s=1/2} \int_{\mathrm{Re}\, t=1/2} h(s,t) E_0(Y_1; s, t) \overline{E_0(Y_2; s, t)}\, ds\, dt \\ &- \sum_{j=1}^{\infty} \frac{1}{2\pi i} \int_{\mathrm{Re}\, s=1/2} h(s, \nu_j) E_1(Y_1; s, \nu_j) \overline{E_1(Y_2; s, \nu_j)}\, ds \\ &- \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\mathrm{Re}\, s=1/2} h(s, \alpha_j) \widetilde{E}_0(Y_1; s, \alpha_j) \overline{\widetilde{E}_0(Y_2; s, \alpha_j)}\, ds \\ &- \sum_{j=1}^{n} \sum_{i=1}^{m} \gamma(\alpha_j, \beta_i) h(\alpha_j, \beta_i) \end{aligned}$$

where $\widetilde{E}_0(Y_1; s, \alpha_k) = \mathrm{Res}_{t=\alpha_k(s)} E_0(Y_1; s, t)$ and

$$\gamma(\alpha_j, \beta_i) = \mathrm{Res}_{s=\beta_i} \widetilde{E}_0(Y_1; s, \alpha_k).$$

So the α_j, β_i are various poles of E_0 and $\widetilde{E}_0$ respectively, and span $\Theta_0^{(1)}$.

Mercer's theorem (see, for example, Courant and Hilbert [5]) then allows us to expand

$$\tilde{k}(Y_1, Y_2) = \sum_{\lambda_1, \lambda_2} h(\lambda_1, \lambda_2) \phi_{\lambda_1, \lambda_2}(Y_1) \overline{\phi_{\lambda_1, \lambda_2}(Y_2)}$$

where the $\phi_{\lambda_1, \lambda_2}$ are an orthonormal basis for the discrete spectrum, indexed by eigenvalues (λ_1, λ_2) of Δ_1 and Δ_2. If we set $Y_1 = Y_2$ and integrate we obtain the trace of $L_{\tilde{k}}$ as

$$\mathrm{tr}\, L_{\tilde{k}} = \sum_{\lambda_1, \lambda_2} h(\lambda_1, \lambda_2) = \int_{\mathfrak{Z}} \tilde{k}(Y, Y)\, d_\mu(Y). \tag{17}$$

The bulk of the calculation of the trace formula goes to show that the right-hand side of this identity converges, rendering the left-hand side an honest trace. We do this by calculating the right-hand side of (16) with a truncation parameter inserted whenever there is a convergence problem, and then letting the truncation parameter tend to infinity. Some of the terms present no convergence problems and these are discussed in the next section.

2. Leading terms

The first noticeable thing about the integral $\int_{\mathfrak{Z}} \tilde{k}(Y, Y)\, d_\mu(Y)$ is that the integrand is the sum of various functions, most of which cannot be integrated separately. One of the summands of $\tilde{k}$ is the original kernel

$$k(Y, Y) = \sum_{\gamma\in\Gamma} f(Y^{-1}\gamma Y)$$

where f is a function in $C_c^\infty(G//\mathbb{K})$. Although the integral of this entire sum is not finite, the elements of Γ can be split into two sets, Γ_1 and Γ_2 where the sum over Γ_1 and the integral over $\mathfrak{J}$ can be interchanged to give a finite number. Γ_1 consists of the identity element and all elements of Γ which are not conjugate to an element of P_1. Γ_2 consists of all elements conjugate to an element of P_1 except the identity element. We must now compute

$$\int_{\mathfrak{J}} \sum_{\gamma\in\Gamma_1} f(Y^{-1}\gamma Y)\, d_\mu(Y) \tag{18}$$

to see that it converges. Recall that $P_1 = P(M_1) \cap SL(3, \mathbb{Z})$ and hence consists of all elements of the form

$$\left(\begin{array}{cc|c} & & a \\ \multicolumn{2}{c|}{M} & b \\ \hline 0 & 0 & \pm 1 \end{array}\right) \quad \text{where } M \in GL(2, \mathbb{Z}),\ a, b \in \mathbb{Z}.$$

So a necessary criterion for an element, $\gamma \in SL(3, \mathbb{Z})$ to be conjugate to an element of P_1 is that it have at least one eigenvalue of ± 1. It turns out that this criterion is also sufficient, that is, every element with an eigenvalue of ± 1 is conjugate under $SL(3, \mathbb{Z})$ to an element of P_1. See Wallace [27] for a proof of this. So Γ_1 consists of the identity element, elements whose characteristic polynomial has three distinct real roots none of which are ± 1 which we will call hyperbolic, and elements whose characteristic polynomial has two imaginary and one real root not equal to ± 1 which we call loxodromic. The identity term is easy to compute.

$$\int_{\mathfrak{J}} f(Y^{-1}IY)\, d_\mu(Y) = \int_{\mathfrak{J}} f(I)\, d_\mu(\gamma) = (\operatorname{vol}\mathfrak{J}) f(I). \tag{19}$$

For the hyperbolic term we wish to compute

$$\int_{\mathfrak{J}} \left(\sum_{\substack{\gamma\in\Gamma \\ \gamma \text{ hyperbolic}}} f(Y^{-1}\gamma Y) \right) d_\mu(Y) = \int_{\mathfrak{J}} \sum_{\{\gamma\}} \sum_{\gamma\in\{\gamma\}} f(Y^{-1}\gamma Y)\, d_\mu(Y) \tag{20}$$

where $\{\gamma\}$ denotes the conjugacy class containing γ

$$= \int_{\mathfrak{J}} \sum_{\{\gamma\}} \sum_{\Gamma\in Z_\gamma\backslash\Gamma} f(Y^{-1}\sigma^{-1}\gamma\sigma Y)\, d_\mu(Y)$$

where Z_γ = centralizer of γ in Γ. Now since the map $Y \to Y^{-1}\gamma Y$ is a dilation for hyperbolic γ we know that only a finite number of the terms

$$\left(\sum_{\gamma\in\{\gamma\}} f(Y^{-1}\gamma Y) \right)$$

are nonzero. Hence the first sum and integral can be interchanged to give

$$\sum_{\{\gamma\}} \int_{\mathfrak{I}} \sum_{\sigma \in Z_\gamma \backslash \Gamma} f(Y^{-1}\sigma^{-1}\gamma\sigma Y)\, d_\mu(Y).$$

Now the map $Y \to \sigma Y$ is discrete so that only a finite number of terms in the summand are nonzero. Thus (20) equals

$$\sum_{\{\gamma\}} \sum_{\sigma \in Z_\gamma \backslash \Gamma} \int_{\mathfrak{I}} f(Y^{-1}\sigma^{-1}\gamma\sigma Y)\, d_\mu(Y).$$

A change of variables gives

$$\sum_{\{\gamma\}} \sum_{\sigma \in Z_\gamma \backslash \Gamma} \int_{\sigma^{-1}\mathfrak{I}} f(Y^{-1}\gamma Y)\, d_\mu(Y)$$

and grouping the $\sigma^{-1}\mathfrak{I}$ together yields

$$\sum_{\{\gamma\}} \int_{\mathfrak{I}_\gamma} f(Y^{-1}\gamma Y)\, d_\mu(Y)$$

where $\mathfrak{I}_\gamma$ is a fundamental region for the centralizer of γ in Γ. We can change variables to conjugate γ in $SL(3, \mathbb{R})$ and take a representative $\tilde{\gamma}$ which is diagonal. Written out in a choice of coordinates, the integrand is given by

$$f(Y^{-1}\tilde{\gamma}Y) = f\left(\begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & 1/\varepsilon_1\varepsilon_2 \end{pmatrix} \begin{pmatrix} 1 & x_1^* & x_2^* \\ 0 & 1 & x_3^* \\ 0 & 0 & 1 \end{pmatrix}\right)$$

and $\{\varepsilon_1, \varepsilon_2, 1/\varepsilon_1\varepsilon_2\}$ are the eigenvalues of γ and

$$\begin{aligned} x_2^* &= x_1\left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1}\right), \\ x_2^* &= x_2\left(\frac{\varepsilon_1 - \varepsilon_3}{\varepsilon_1}\right) + x_1x_3\left(\frac{\varepsilon_3 - \varepsilon_2}{\varepsilon_1}\right), \\ x_3^* &= x_3\left(\frac{\varepsilon_2 - \varepsilon_3}{\varepsilon_2}\right), \\ Y &= \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & 1/y_1y_2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

and

$$d_\mu(Y) = \frac{dy_1\, dy_2}{y_1y_2}\, dx_1\, dx_2\, dx_3.$$

It remains to describe the nature of Z_γ, $\mathfrak{I}_\gamma$, and $\{\gamma\}$. It follows from [23] that Z_γ is isomorphic to the group of units in the order $\mathbb{Z}[\varepsilon]$ for ε any eigenvalue of any $\gamma \in Z_\gamma$, $\gamma \pm I$. $\mathfrak{I}_\gamma$ is therefore a strip, $-\infty < x_1 < \infty$, $-\infty < x_2 < \infty$, $-\infty < x_3 < \infty$, whose cross section in the (y_1, y_2) plane has area equal to the regulator of $\mathbb{Z}[\varepsilon]$. The number of conjugacy classes $\{\gamma\}$ with eigenvalues $\{\varepsilon_1, \varepsilon_2, 1/\varepsilon_1\varepsilon_2\}$ is the class number of the ring $\mathbb{Z}[\varepsilon]$. With these comments and the preceding changes of variables, (17) becomes

$$\sum_{\substack{\text{totally real cubic}\\ \text{number fields}\\ \text{with fundamental}\\ \text{units } \alpha_1,\beta_1}} \sum_{j,k\neq(0,0)} \mathrm{cl}(\mathbb{Z}[\alpha_1])\,\mathrm{Re}\,g(\mathbb{Z}[\alpha_1]) \left|\frac{(\alpha_1^j\beta_1^k)^2(\alpha_2^j\beta_2^k)}{W(\alpha_1^j\beta_1^k)}\right| \hat{\hat{f}}(A^jB^k) \tag{21}$$

where

$$A=\begin{pmatrix}\alpha_1 & & \\ & \alpha_2 & \\ & & 1/\alpha_1\alpha_2\end{pmatrix}, \qquad B=\begin{pmatrix}\beta_1 & & \\ & \beta_2 & \\ & & 1/\beta_1\beta_2\end{pmatrix},$$

$W(\alpha_1^j\beta_1^k) = (\alpha_1^j\beta_1^k - \alpha_2^j\beta_2^k)(\alpha_2^i\beta_2^k - \alpha_3^j\beta_3^k)(\alpha_1^j\beta_1^k - \alpha_3^j\beta_3^k)$, $\mathrm{cl}(R)$ is the class number of R, and $\mathrm{Re}\,g(R)$ is the regulator of R. Also $\hat{f}(E)$ is the transform of f at a diagonal element E given by

$$\hat{\hat{f}}(E)=\int_N f(EN)\,dN.$$

This transform has various names. See Lang [15], Terras [23], or Hejhal [9]. We will call it the Harish-Chandra transform.

The geometric interpretation of this term resides in the Z_γ. Because the centralizer of γ has two generators the fundamental region for it in some copy of A is a torus with area $\mathrm{Re}\,g(\mathbb{Z}[\alpha_1])$. Because it sits in a copy of A, it is a totally geodesic (flat) submanifold of $\mathfrak{I}$. Furthermore it turns out that the hyperbolic elements are the only ones which give rise to flat tori in $\mathfrak{I}$. That is, these are the only elements whose centralizers are both confined to a split Cartan subgroup of G and have two generators. So the hyperbolic term counts all the flat tori which occur in $\mathfrak{I}$.

For the loxodromic terms, we have

$$\int_{\mathfrak{I}} \sum_{\substack{\gamma\in\Gamma\\ \gamma \text{ loxodromic}}} f(Y^{-1}\gamma Y)\,d_\mu(Y)$$

and for the same reasons as for the hyperbolic term we can interchange sums and integrals so that (19) equals

$$\sum_{\{\gamma\}}\sum_{\gamma\in\{\gamma\}}\int_{\mathfrak{I}} f(Y^{-1}\gamma Y)\,d_\mu(Y) \tag{22}$$

where $\{\gamma\}$ refers to the conjugacy class with representative γ.

Now, γ is the element of some Cartan subgroup of $SL(3,\mathbb{R})$ which is not split over $\mathbb{R}$. In other words, γ is conjugate under $SL(3,\mathbb{R})$ to

$$\tilde{\gamma}=\left(\begin{array}{cc|c} & & 0\\ r & R_\theta & 0\\ \hline 0 & 0 & r^{-2}\end{array}\right) \tag{23}$$

where

$$R_\theta=\begin{pmatrix}\cos\theta & \cos\theta\\ \sin\theta & \cos\theta\end{pmatrix}.$$

Furthermore, the generator of $Z(\gamma)$ goes, under the same change of variables, to

$$\tilde{\gamma}_0 = \left(\begin{array}{cc|c} & & 0 \\ r_0 & R_{\theta_0} & 0 \\ \hline 0 & 0 & r_0^{-2} \end{array}\right).$$

A fundamental region for the action of $\tilde{\gamma}_0$ in the coordinates given earlier is $\mathfrak{I}_{\tilde{\gamma}} = \{Y | 1 \le y_1 < r_0,\ u_1 > 0,\ x_1,\ t_1,\ v_1 \in \mathbb{R}\}$.

Furthermore, the conjugacy classes are arranged by cubic number field with multiplicity equal to the class number of an order. See [23] for more details. We denote the class number associated to $\mathbb{Z}[r_0 e^{i\theta_0}]$ as $\mathrm{cl}(r_0, \theta_0)$.

Writing Y and $\tilde{\gamma}$ as in (23) and doing some arithmetic to the formula in (22) yields

(24)

$$\sum_{\lambda} \mathrm{cl}(r_0, \theta_0) \int_{\mathfrak{I}_{\tilde{\gamma}}} f\left(\left(\begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^{-2} \end{pmatrix}\right)\left(\begin{array}{cc|c} X^{-1} & R_\theta X & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\begin{pmatrix} 1 & 0 & x^* \\ 0 & 1 & t^* \\ 0 & 0 & 1 \end{pmatrix}\right) d_\mu(Y)$$

where

$$X = \begin{pmatrix} u_1^{1/2} & u_1^{-1/2} v_1 \\ 0 & u_1^{-1/2} \end{pmatrix}$$

and

$$\begin{pmatrix} x^* \\ t^* \end{pmatrix} = (I_{2x2} - r^{-3}(X^{-1} R_\theta X)) \begin{pmatrix} x_1 \\ t_1 \end{pmatrix}.$$

In these coordinates, $d_\mu(Y) = dx\,dt\,dv(du/u^2)(dy/y)$. The determinant of the Jacobian of this transformation is given by

$$|J| = |1 - r^{-3}(2\cos\theta) + r^{-6}|.$$

Making the change of variables and integrating out y_1 leaves (24) in the form

(25)

$$\sum_{\Lambda} \frac{\mathrm{cl}(r_0, \theta_0) |\ln r_0|}{|1 - wr^{-3}\cos\theta + r^{-6}|}$$
$$\times \int_{u>0} \iiint_{v,x,t \in \mathbb{R}^3} f\left(\begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^{-2} \end{pmatrix}\left(\begin{array}{cc|c} X^{-1} & R_\theta X & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\right) dx\,dt \frac{du\,dv}{u^2}.$$

Here we have written out the remaining part of $d_\mu(Y)$ and we note that if we regard X as the point in the Poincaré upper half-plane, $\mathbb{H}$, given by $v + iu$, then $(1/u^2)\,dv\,du$ is the $SL(2, \mathbb{R})$ invariant measure associated to $\mathbb{H}$ in the usual way.

The transform given by

$$\hat{f}(Y) = \int_{x \in \mathbf{R}} \int_{t \in \mathbf{R}} f\left(Y \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\right) dx\,dt \tag{26}$$

is a transform which occurs other places in the Selberg trace formula for this group and is also a type of Harish-Chandra transform. The integrand in (25) is equal to

$$\hat{f}\left(\begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^{-2} \end{pmatrix}\left(\begin{array}{cc|c} X^{-1} & R_\theta X & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\right)$$

after integrating out x and t. $\hat{\phi}$ is $SO(2, \mathbb{R})$ bi-invariant in the coordinates $v + iu \in \mathbb{H}$ with the usual action.

A slight modification of the argument in [30] on page 12 tells us that

$$\int_{iu+v} \hat{f}\left(\begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^{-2} \end{pmatrix} \begin{pmatrix} X^{-1} & R_\theta X & \\ & & \\ & & 1 \end{pmatrix}\right) u^t \frac{du\,dv}{u^2}$$

$$= \left(\int_{\mathrm{Re}\,s=1/2} h_\theta(s, t) r^{1-s}\, ds\right) u^t$$

where h_θ is the Selberg transform of f_θ and where

$$f_\theta(r, u, v, x, t) = f(r, u', v', x, t)$$

and

$$u' = \mathrm{Im}((X^{-1} R_\theta X) \circ i), \qquad X = \begin{pmatrix} \sqrt{u} & v/\sqrt{u} \\ 0 & 1\sqrt{u} \end{pmatrix},$$

$$v' = \mathrm{Re}((X^{-1} R_\theta X) \circ i).$$

Combining this with a calculation due to Selberg and found in Kubota [14] we can write the orbital integral in (25) as

$$(27) \qquad \int_{\mathrm{Re}\,t=1/2} \int_{\mathrm{Re}\,s=1/2} h(s, t) \frac{r^{1-s} e^{-2\theta t}}{1 + e^{-2\pi t}}\, ds\, dt.$$

Writing the pair (r, θ) as powers of r_0 and multiples of $\theta_0 \bmod \pi$, we can write the loxodromic term as

$$(28) \qquad \sum_{\substack{\mathbb{Z}[r_0 e^{i\theta_0}] \\ \text{distinct} \\ \text{mixed cubic} \\ \text{number fields}}} \sum_{j>0} \frac{\mathrm{cl}(r_0, \theta_0) |\ln r_0|}{|1 - 2r_0^{-3j} \cos j\theta + r_0^{-6j}|} \times \int_{\mathrm{Re}\,t=1/2} \int_{\mathrm{Re}\,s=1/2} h(s, t) \frac{r_0^{j(1-s)} e^{-2j\theta_0 \,\mathrm{Im}\, t}}{1 + e^{-2\pi t}}\, ds\, dt.$$

Note that $|\ln r_0|$ is the regulator of the number field. Together formulas (19), (21), and (28) account for all the orbital integrals for Γ_1, and together they comprise the leading terms of the trace formula. We refer the reader to [23, 24, 26, and 29] for further details about the calculations in this section.

3. Introduction to the parabolic term

As pointed out earlier, $P(M_1)$ is a parabolic subgroup and M_1 is a reductive subgroup of G. Recall that M_1 is

$$\left\{ \left(\begin{array}{cc|c} * & * & 0 \\ * & * & 0 \\ \hline 0 & 0 & \pm 1 \end{array}\right) \in G \right\}$$

and so has an Iwasawa decomposition of its own given by

$$\begin{pmatrix} u^{1/2} & u^{-1/2} v & 0 \\ 0 & u^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \left(\begin{array}{cc|c} \widetilde{K} & 0 & \\ & & \partial \\ \hline 0 & 0 & \pm 1 \end{array}\right)$$

where $\widetilde{K}$ is isomorphic to $O(2, \mathbb{R})$. So sitting inside $P(M_1)$ is a copy of $\mathbb{H}$, the Poincaré upper half plane, represented as $M_1/\widetilde{K}$. Now $P_1 = P(M_1) \cap SL(3, \mathbb{Z})$ acts on $P(M_1)$ and has a subgroup, $M_1 \cap SL(3, \mathbb{Z})$, which is isomorphic to $GL(2, \mathbb{Z})$ and it acts on $M_1/\widetilde{K}$ to give a fundamental region, $\mathfrak{I}_2$ which is half of the standard one for $SL(2, \mathbb{Z})$.

The strategy for computing the parabolic term for $SL(3, \mathbb{Z})$ is to reduce all orbital integrals and Eisenstein series, whenever possible, to the ones that occur in the trace formula for $GL(2, \mathbb{Z})$ acting on $\mathbb{H} = M_1/\widetilde{K}$. This rank one formula will involve a particular $\widetilde{K}$-bi-invariant convolution kernel, given by

$$\tilde{f}(v + iu) = \iint_{x, t \in \mathbb{R}} f\left(\begin{pmatrix} u^{1/2} & uv^{-1/2} & 0 \\ 0 & u^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \right) dx\, dt. \tag{29}$$

This is a type of Harish-Chandra transform, equivalent (up to notation) to formula (26), and the reader will notice that it is $\widetilde{K}$-bi-invariant. It has an associated Selberg transform which we will call $\check{h}(t)$ and we have shown in [28] that

$$\check{h}(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} s = 1/2} h(s, t)\, ds \tag{30}$$

where h is the Selberg transform associated to f.

There are a few differences between the trace formula for $GL(2, \mathbb{Z})$ and for $SL(2, \mathbb{Z})$. The major one is that only the even cusp forms for $SL(2, \mathbb{Z})$ are cusp forms for $GL(2, \mathbb{Z})$. Fortunately only even cusp forms show up in the construction of Eisenstein series for $SL(3, \mathbb{Z})$. Another detail which may cause confusion is that they are normalized to reflect the fact that $\mathfrak{I}_2$ is only half as large as the fundamental region for $SL(2, \mathbb{Z})$. So mysterious factors of 2 appear at certain places in our calculations.

Finally we must warn the reader that not all orbital integrals and Eisenstein series can be reduced to the pattern described here. That is why there is a rank zero cancellation as well as a rank one cancellation. In the next four sections we summarize the content of several papers devoted to the calculation of this term.

4. Parabolic orbital integrals

These integrals fall into two broad groups: those which come from elements in P_0 and the rest. The elements of P_0 are those with more than one eigenvalue equal to ± 1, where as the others have exactly one eigenvalue equal to ± 1.

First we look at the case where exactly one eigenvalue is ± 1. For more details than are given here, see [26 and 30]. Any such element can be conjugated into P_1 and has a centralizer which is cyclic. In [26] we made an attempt to count the conjugacy classes corresponding to eigenvalues λ, $\tilde{\lambda}$, ± 1, but only obtained a bound on the number. Similarly we tried to compute the centralizer explicitly and got a partial description of that also. To finish the trace formula these descriptions must be exact, and we give them here, referring the reader to [26] for parts of the proofs.

Since we are in P_1 we use coordinates given by (2) with $y_1 = \gamma_1^6$. In these coordinates we have a fundamental region with a cusp at $y_1 = +\infty$ and $u_1 =$

$+\infty$. We will have to truncate the region $\mathfrak{I}$ to have these various terms converge separately. We cut it off at $y_1 = T$ (and $u_1 = S$ if necessary).

If our representative of a conjugacy class is

$$\gamma = \left(\begin{array}{cc|c} M & & a \\ & & b \\ \hline 0 & 0 & \pm 1 \end{array}\right)$$

then its centralizer is of the form

$$\gamma_0 = \left(\begin{array}{cc|c} M_0 & & c \\ & & d \\ \hline 0 & 0 & \pm 1 \end{array}\right)$$

where $M = M_0^k$ for some k. Suppose $|\operatorname{tr} M| > 2$. We can change coordinates to conjugate both of these to Jordan normal form where

$$\tilde{\gamma} = \begin{pmatrix} \varepsilon^k & 0 & 0 \\ 0 & \tilde{\varepsilon}^k & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}, \qquad \tilde{\gamma}_0 = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \tilde{\varepsilon} & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

Obviously $\tilde{\varepsilon} = \pm\varepsilon^{-1}$. It is easy to see that a fundamental region for $\tilde{\gamma}_0$ is the cylinder between $u_1 = 1$ and $u_1 = \varepsilon^2$. The calculation for the integral is essentially the same as for the hyperbolic case, and we get

(31)

$$\sum_{\{\varepsilon^k, \tilde{\varepsilon}, \pm 1\}} \sum_{\substack{\{\delta_i(\varepsilon^k, \tilde{\varepsilon}^k, \pm 1)\} \\ \text{conjugacy classes} \\ \text{with these} \\ \text{eigenvalues}}} \int_{x_1} \int_{t_1} \int_{v_1} \int_{y_1 = 1/T}^{T} \int_{u_1 = 1}^{\varepsilon^2 j(k_i)} f\left(g^{-1} \begin{pmatrix} \varepsilon^k & & \\ & \tilde{\varepsilon}^k & \\ & & \pm 1 \end{pmatrix} g\right) dgk.$$

where

$$g = \begin{pmatrix} y_1^6 & 0 & 0 \\ 0 & y_1^6 & 0 \\ 0 & 0 & y_1^{-12} \end{pmatrix} \begin{pmatrix} u_1^{1/2} & u_1^{-1/2} & 0 \\ 0 & u_1^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & v_1 & x_1 \\ 0 & 1 & t_1 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$dgK = \frac{dy_1\, dy_1}{u_1 y_1}\, dx_1 k t_1\, dv_1, \qquad \gamma_1 = y_1^6,$$

and $j(k)$ is the power of ε generating the centralizer of $\gamma(\varepsilon^k, \tilde{\varepsilon}^k, \pm 1)$.

Expanding (31) in these coordinates, multiplying and rewriting the variable in coordinates of type (2) gives

$$\begin{aligned} &\sum_{\{\varepsilon^k, \tilde{\varepsilon}^k, \pm 1\}} \xi(\varepsilon^k, \tilde{\varepsilon}^k, \eta(\varepsilon^k)) \cdot \frac{2 \ln T \ln \varepsilon^2 \varepsilon^k}{(\varepsilon^k - \tilde{\varepsilon}^k)(\varepsilon^k - \eta(\varepsilon^k))(\eta(\varepsilon^k) - \tilde{\varepsilon}^k)} \\ &\times \int_{v \in \mathbb{R}} \hat{f}\left(\begin{pmatrix} \varepsilon^k & 0 & 0 \\ 0 & \tilde{\varepsilon}^k & 0 \\ 0 & 0 & \eta(\varepsilon^k) \end{pmatrix} \begin{pmatrix} 1 & v_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) dv_1 \end{aligned} \tag{32}$$

where $\eta(\varepsilon^k) = \pm 1$ as appropriate, and

$$\xi(\varepsilon^k, \tilde{\varepsilon}^k, \eta(\varepsilon^k)) = \sum_{\{\delta_i(\varepsilon^k, \tilde{\varepsilon}^k, \eta(\varepsilon^k))\}} k_i.$$

The pertinent fact missing from [27] is that

$$\xi(\varepsilon^k, \tilde{\varepsilon}^k, \eta(\varepsilon^k)) = \begin{cases} |\operatorname{tr} M - 2| \cdot \operatorname{cl}(\varepsilon, \tilde{\varepsilon}) & \text{if } \eta = 1, \\ |\operatorname{tr} M| \cdot \operatorname{cl}(\varepsilon, \tilde{\varepsilon}) & \text{if } \eta = -1, \end{cases} \tag{33}$$

where $\operatorname{cl}(\varepsilon, \tilde{\varepsilon})$ is the usual class number in $GL(2, \mathbb{Z})$ of M. The proof of this fact is in Appendix 1. Then (33) allows us to rewrite (32) as

$$2 \ln T \left[\sum_{\substack{\mathbb{Z}[\varepsilon] \\ \text{orders in} \\ \text{real quadratic} \\ \text{number fields}}} \sum_{k>0} \frac{\ln \varepsilon^2 \varepsilon^k}{\varepsilon^k - \tilde{\varepsilon}^k} \int_{v \in \mathbb{R}} \hat{f}\left(\begin{pmatrix} \varepsilon^k & 0 \\ 0 & \tilde{\varepsilon}^k \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \right) dv \right]. \tag{34}$$

The reader will recognize the part in brackets as the hyperbolic term for the trace formula for $\mathrm{GL}(2, \mathbb{Z})$ acting on $\mathbb{H}$. There is no need to truncate in the variable u since this term converges.

Similarly for the elliptic M, we get

$$2 \ln T \left[\sum_{\{\gamma\} \text{ elliptic}} \frac{1}{2M \sin(l\pi/M)} \int_{-\infty}^{\infty} \frac{e^{-(2l\pi/M)t}}{1 + e^{-2\pi t}} \check{h}(t)\, dt \right] \tag{35}$$

where the part in brackets is the usual elliptic term for $GL(2, \mathbb{Z})$. This of course depends on a result similar to (33), also found in Appendex 1.

Now we need to consider the elements of P_0. For the purposes of computation they fall into four classes.

I.

$$\left\{ \begin{pmatrix} 1 & \tilde{a}d & b \\ 0 & 1 & \tilde{c}d \\ 0 & 0 & 1 \end{pmatrix}, \ \begin{matrix} d \neq 0 \\ \tilde{a}, b, \tilde{c}, d \in \mathbb{Z} \end{matrix}, \ (\tilde{a}, \tilde{c}) = 1 \right\}$$

and all γ conjugate to one of these,

II. those conjugate to

$$\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b \in \mathbb{Z},$$

III. γ conjugate in $SL(3, \mathbb{R})$ to

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

IV. γ conjugate to one of

$$\left\{ \begin{pmatrix} -1 & a & b \\ 0 & -1 & c \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

but not elliptic. All of these are computed in [27], and in more detail than here. We present a quick description of each case.

I. The centralizer of γ consists of elements of the form

$$\begin{pmatrix} 1 & \tilde{a}n & b \\ 0 & 1 & \tilde{c}n \\ 0 & 0 & 1 \end{pmatrix}.$$

Using coordinates of the form

$$Y = \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & 1/y_1y_2 \end{pmatrix} \tag{36}$$

a fundamental region for the centralizer of γ is given by

$$\begin{cases} 0 < y_1 < \infty, \quad 0 < y_2 < \infty, \quad 0 < x_2 < 1, \\ 0 < x_1 < \tilde{a}, \quad -\infty < x_3 < \infty. \end{cases} \tag{37}$$

The representatives of the conjugacy classes of elements in I are

$$\begin{pmatrix} 1 & a & b \pmod{n} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \qquad (a, c) = n, \ a, b, c > 0,$$

and

$$\begin{pmatrix} 1 & -a & -b \pmod{n} \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix},$$

Changing the y_1, y_2 coordinates to the y_1, u_1 coordinates of (2) and truncating y_1 and u_1 gives the sum over elements of this type to be

$$2\sum_{a>0}\sum_{c>0}\sum_{b=1}^{n=(a,c)} \int_{y=1/T}^{T}\int_{u=1/S}^{S}\int_{x_2=0}^{1}\int_{x_3=-\infty}^{\infty}\int_{x_1=0}^{a/n} \times f\left(\begin{pmatrix} 1 & a/u & u^{-1/2}y^{-3}(ax_3 - cx_1 + b) \\ 0 & 1 & cy^{-3}u^{1/2} \\ 0 & 0 & 1 \end{pmatrix}\right) \frac{dx_1\,dx_2\,dx_3\,dy_1\,du_1}{u_1^2 y_1^7}.$$

After suitable changes of variables and integrating out x_1 and x_2 this becomes

$$2\sum_{a>0}\int_{u=1/S}^{S}\int_{v=-\infty}^{\infty}\sum_{c>0}\int_{w=cu^{1/2}T^{-3}}^{cu^{1/2}T^3} f\left(\begin{pmatrix} 1 & a/u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}\right) \frac{dw\,dv\,du}{3cu^2}. \tag{39}$$

As in the classical trace formula, we interchange the inner sum and integral, neglect terms which are $o_T(1)$ and (39) becomes

$$4\ln T\sum_{a>0}\int_{u=1/S}^{S}\int_{v=-\infty}^{\infty}\int_{w=0}^{\infty} f\left(\begin{pmatrix} 1 & a/u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}\right) \frac{du}{u^2}\,dv\,dw$$

which in turn equals

$$2\ln T\sum_{a>0}\int_{u=1/S}^{S} \hat{f}\left(\begin{pmatrix} 1 & a/u \\ 0 & 1 \end{pmatrix}\right) \frac{du}{u^2}. \tag{40}$$

The reader will recognize this as the parabolic orbital integrals of the type $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ for $GL(2, \mathbb{Z})$, for the kernel obtained from $\hat{f}$.

II. Recall that this term consists of elements of the form

$$\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b \in \mathbb{Z},\ b \neq 0,$$

and we see immediately that the centralizer is all of P_0. Four copies of the fundamental region for P_0 fit into the strip

$$\begin{gathered} 0 - \infty < y_1 < \infty, \quad -\infty < u_1 < \infty, \\ 0 < x_1 < 1, \quad 0 < x_2 < 1, \quad 0 < x_3 < 1. \end{gathered} \tag{41}$$

The orbital integral in these coordinates is given by

$$\frac{1}{4} \sum_{b>0} \int_{u_1=1/S}^{S} \int_{y_1=1/T}^{T} \int_{x_1=0}^{1} \int_{x_2=0}^{1} \int_{x_3=0}^{1} f\left(\begin{pmatrix} 1 & 0 & u_1^{-1/2} y_1^{-3} b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \cdot \frac{du}{u_1^2 a} \frac{dy_1}{y_1^7} dx_1\, dx_2\, dx_3. \tag{42}$$

Setting $v = u^{-1/2} y^{-3} b$ and changing the integral with respect to u gives

$$\frac{1}{4} \int_{y=1/T}^{T} \sum_{b>0} \int_{v=S^{-1/2}y^{-3}b}^{S^{1/2}y^{-3}b} f\left(\begin{pmatrix} 1 & 0 & v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \frac{2v\, dv\, dy}{b^2 y}. \tag{43}$$

Interchanging sum and integral in (43) gives

$$\frac{1}{4} \cdot 2 \cdot \zeta(2) \int_{v=0}^{\infty} \int_{y=1/T}^{T} f\left(\begin{pmatrix} 1 & 0 & v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) v\, dv \frac{dy}{y}$$

and integrating out y gives

$$\zeta(2) \ln T \int_{v=0}^{\infty} f\left(\begin{pmatrix} 1 & 0 & v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) v\, dv. \tag{44}$$

Equation (44) is an example of a term which does *not* reduce to an integral we recognize as part of the trace formal for $GL(2, \mathbb{Z})$.

III. Here γ is elliptic and is therefore conjugate in $SL(3, \mathbb{Z})$ to either

$$\gamma_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \gamma_2 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For γ_1 the centralizer looks like a copy of $GL(2, \mathbb{R})$ and the calculation of the orbital integral is routine. We get

$$2 \ln T \operatorname{vol} \mathfrak{I}_2 \hat{f} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{45}$$

Of course this is a multiple of the orbital integral for $GL(2, \mathbb{Z})$ for the element $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

For γ_2 the centralizer is conjugate to a congruence subgroup. $\Gamma_0(2)$ of $GL(2, \mathbb{Z})$. Knowing this, the integral is routine and gives

$$2 \ln T \operatorname{vol} \mathfrak{I}(\Gamma_0(2)) \hat{f} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{46}$$

where $\mathfrak{I}(\Gamma_0(2))$ is a fundamental region for the subgroup of $GL(2, \mathbb{Z})$ whose lower left entry is congruent to $0 \bmod 2$.

IV. There are three forms this case covers

$$\gamma_1 \cong \begin{pmatrix} -1 & a & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma_2 \cong \begin{pmatrix} -1 & a & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma_3 \cong \begin{pmatrix} -1 & a & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Each of these is a computation similar to what we have done already and they all give a multiple of the same lower rank integral. We just state that the total contribution is

$$\left(\frac{5}{4}\right) \cdot 2 \ln T \sum_{a>0} \int_{u=1/S}^{S} \hat{f}\begin{pmatrix} -1 & -a/u \\ 0 & -1 \end{pmatrix} \frac{du}{u^2}. \tag{47}$$

This concludes the discussion of the orbital integrals which diverge in the trace formula. After truncation their contribution is given by the sum of formulas (34), (35), (40), (44)–(47). We now turn to a discussion of the various Eisenstein series which contribute divergent integrals to the trace formula.

5. Eisenstein series

We must calculate to the contribution of the Eisenstein series which are subtracted from our kernel in formula (16). First we will look at those which are induced from even cusp forms on $SL(2, \mathbb{Z})$. We are subtracting something of the form

$$\int_{\mathfrak{I}} \frac{1}{2\pi i} \int_{\operatorname{Re} s=1/2} h(s, \nu_j) E_1(Y; s, \nu_j) \overline{E_1(Y; s, \nu_j)}\, ds\, dgK. \tag{48}$$

This term is computed in [28] and we summarize the results here. Replacing $\bar{s}$ by $1-s$ and truncating $\mathfrak{I}$ at $y_1 = T$ we get

$$-\frac{1}{2\pi i} \int_{\operatorname{Re} s=1/2} h(s, \nu_j) \int_{\mathfrak{I}_T} E(Y; s, \nu_j) E(Y; 1-s, \overline{\nu}_j)\, dgK\, ds$$

and we then wish to move the line of integration to $\operatorname{Re} s > 1$. We can do this because any residue of a pole of the product of Eisenstein series will contribute zero to the final formula (see [28]). We are also making use of the Paley-Wiener theorems for the Helgason transform [11]. After moving the contour and unwinding the first of the two Eisenstein series, we get that (48) equals

$$\frac{1}{2\pi i} \int_{\operatorname{Re} s=1+\varepsilon} h(s, \nu_j) \sum_{\gamma \in P_1 \backslash \Gamma} \int_{\gamma \circ \mathfrak{I}_T} \nu_j(v_i + i u_i) y_1^s E_1(Y; 1-s, \overline{\nu}_j)\, dY\, ds. \tag{49}$$

A messy estimate shows we can replace the region of integration

$$\int_{\bigcup_{\gamma \in P_1 \backslash \Gamma} \gamma \circ \mathfrak{I}_T} \quad \text{by the strip} \quad \frac{1}{4} \int_{y_1=1/T}^{T} \int_{v_1 + i u_1 \in \mathfrak{I}_3} \int_{x_1=0}^{1} \int_{t_1=0}^{1}$$

where $\mathfrak{I}_3$ is a fundamental region for $SL(2, \mathbb{Z})$. This estimate is in [28] or alternatively, see Appendix 2.

At this point the only function depending on x_1 and t_1 is the remaining Eisenstein series, $E_1(Y; 1-s, \overline{\nu}_j)$. After integrating we are left with the constant term in its Fourier expansion in t_1. Thus (49) becomes

$$\begin{aligned}(49a)\qquad &\frac{1}{4}\frac{-1}{2\pi i}\int_{\mathrm{Re}\, s=1+\varepsilon} h(s, \nu_j)\int_{y_1=1/T}^{T}\int_{v_1+iu_1\in\mathfrak{I}_3}\int_{x_1=0}^{1} y_1^s\nu_j(v_1+iu_1)\\ &\times[y_1^{1-s}\overline{v}_j h(v_1+iu_1)+y_2^{1-s}c_2(\overline{v}_j, 1-s)(z_2)]\frac{dx_1\,dy_1\,du_1\,dv_1}{y_1^2u_1^2}\,ds.\end{aligned}$$

The second term in the summand has as a factor the function $c_2(\overline{\nu}_j, 1-s)(z_2)$ which is some cusp form in the variable z_2. Integrating out x_2 gives zero as its integral, for reasons given in [28]. So (49a) becomes

$$-\frac{1}{2\pi i}\int_{\mathrm{Re}\, s=1+\varepsilon} h(s, \nu_j)\frac{1}{4}\int_{y_1=1/T}^{T}\int_{v_1+iu_1\in\mathfrak{I}_3} y_1\|\nu_j(v_1+iu_1)\|^2\frac{dy_1}{y_1^2}\frac{dv_1\,du_1}{u_1^2}\,ds.$$

The ν_j are normalized so that their square integral over $\mathfrak{I}_3$ is 2, so doing the remaining integrals gives a contribution of $2\ln T\check{h}(\nu_j) = 2\ln T\check{h}(t_j)$, for this term. Altogether, the contribution from these types of Eisenstein series is

$$(50)\qquad -2\ln T\sum_{v_j}\check{h}(t_j).$$

Next we turn to those Eisenstein series which are built out of the functions $y_1^s u_1^t$. References for this calculation include [28, 12, and 33].

Recall from §1 that these Eisenstein series are constructed as

$$E_0(Y, s, t) = \sum_{\gamma\in P_0\backslash\Gamma} y_1(\gamma Y)^S u_1(\gamma Y)^t$$

for $\mathrm{Re}\, s$ and $\mathrm{Re}\, t$ large. An alternative way to construct these for $\mathrm{Re}\, t$ small is as the sum, (given in Imai and Terras [12]),

$$E_0(Y, s, t) = \sum_{\gamma\in P_1\backslash\Gamma} E(u_1(\gamma Y), t)y_1(\gamma Y)^S,$$

where E is an Eisenstein series for $SL(2, \mathbb{Z})$. The poles of E_0 are given in Figure 1 and the two ways of writing E_0 will help us move the plane of integration across these lines one at a time.

As we cross each of these lines we pick up a residue which is a lower rank Eisenstein series. All three lines are equivalent under the functional equations to $t = 1$ (or 0). Here is the full set of functional equations for these series:

a. $c(1-t)E(Y, s, t) = E(Y, s, 1-t)$,

b. $c\left(1-\frac{3s-t}{2}\right)E(Y, s, t) = E\left(Y, \frac{1}{2}(1-s+t), \frac{1}{2}(-1+3s+t)\right)$,

c. $c(1-t)c\left(1-\frac{3s-t-1}{2}\right)E(Y, s, t) = E\left(Y, 1-\frac{s}{2}-\frac{t}{2}, 1-\frac{3s}{2}+\frac{t}{2}\right)$,

d. $c\left(1-\frac{3s-t}{2}\right)c\left(1-\frac{3s-t-1}{2}\right)E(Y, s, t)$
$= E\left(Y, \frac{1}{2}(1-s+t), \frac{1}{2}(3-3s-t)\right)$,

e. $c(1-t)c\left(1-\frac{3s-t-1}{2}\right)E(Y, s, t) = E\left(Y, 1-\frac{s}{2}-\frac{t}{2}, \frac{3s-t}{2}\right)$.

$t = 1$
$3s - t = 2$
$3s + t = 3$

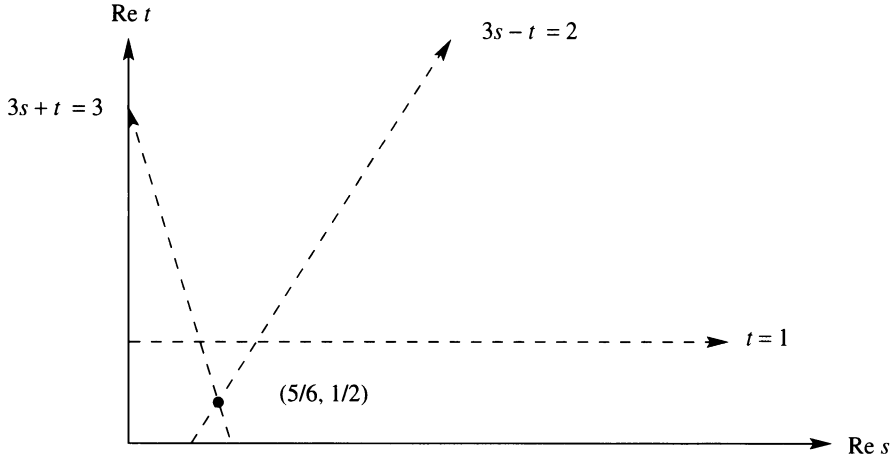

FIGURE 1

Here the function c is given by

$$c(s) = \frac{\pi^{1/2}\Gamma\left(s - \frac{1}{2}\right)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}.$$

As described in formula (16) we must calculate

$$\frac{1}{2\pi i}\int_{\operatorname{Re} s=1/2}\int_{\operatorname{Re} t=1/2} h(s, t)\int_{\mathfrak{Z}_{T,S}} E_0(Y, s, t)\overline{E_0(Y, s, t)}\,dy\,ds\,dt, \tag{52}$$

where $\mathfrak{Z}_{T,S} = \{Y \in \mathfrak{Z} | y(z) < T,\ u(z) < S\}$. For the residues at the poles $s = 1$, etc., we must compute a similar integral.

We compute (52) by replacing $\bar{s}$ and $\bar{t}$ by $1 - s$, $1 - t$, moving the line of integration to $\operatorname{Re} s$, $\operatorname{Re} t$ large, computing there and moving the lines of integration back to $\operatorname{Re} s = \operatorname{Re} t = \frac{1}{2}$. When we move the lines of integration we must now worry about poles of the function

$$g(s, t) = E(Y, s, t)E(Y, 1-s, 1-t). \tag{53}$$

Ignoring the poles of g for the moment, we will compute (52) where $\operatorname{Re} s$ and $\operatorname{Re} t$ are large. We can do an unwinding argument when $\operatorname{Re} s$ is large to replace

$$\int_{\operatorname{Re} s=c_1}\int_{\operatorname{Re} t=c_2}\int_{\mathfrak{Z}_{T,S}} h(s, t)E(Y, s, t)E(Y, 1-s, 1-t)\,d\mu(Y)\,ds\,dt$$

by

$$\int_{\operatorname{Re} s=c_1}\int_{\operatorname{Re} t=c_2}\int_{\bigcup_{\gamma\in P_1\backslash\Gamma}\gamma^{-1}\mathfrak{Z}_{T,S}} y_1^s(E(z_1, t)E(Y, 1-s, 1-t)\,d\mu(Y)\,ds\,dt$$

and an estimate which is done in Appendix 2 shows that the integral above may be replaced by

$$\int_{\mathrm{Re}\, s=c_1}\int_{\mathrm{Re}\, t=c_2}\int_{0<x_1\,,\,x_2<1}\int_{1/T<y<T}\int_{\mathfrak{I}_{3,S}} y_1^s E(z_1\,,\,t) \times E(Y\,,\,1-s\,,\,1-t)\,d\mu(z_1)\frac{dx_1\,dt_1\,dy_1}{y_1^2}\,ds\,dt$$

where $\mathfrak{I}_{3,S}$ is the fundamental region for $SL(2,\mathbb{Z})$ acting on $z_1 = v_1 + iu_1$, truncated at $u_1 < B$, and $d\mu(z_1)$ is the $SL(2,\mathbb{R})$ invariant measure for z_1.

Integrating over x_1 and t_1 we are left with the constant term in the Fourier expansion of $E(Y\,,\,1-s\,,\,1-t)$ which (due to Venkov [25]) is

$$\begin{aligned} y_1^{1-s}E(z_1\,,\,1-t) + y_1^{\frac{1}{2}(1+s-t)}c\left(\frac{2-3s+t}{2}\right)E\left(z_1\,,\,\frac{3-3s-t}{2}\right) \\ + y_1^{\frac{s+t}{2}}c(1-t)c\left(\frac{3-3s-t}{2}\right)E\left(z_1\,,\,\frac{2-3s+t}{2}\right) \end{aligned} \tag{54}$$

where $z_1 = v_1 + iu_1$ and $E(z_1\,,\,*)$ is an Eisenstein series for $SL(2,\mathbb{Z})$.

The first term in the summand (54) gives

$$\int_{\mathrm{Re}\, s=c_1}\int_{=\mathrm{Re}\, t=c_2}\int_{z_1\in\mathfrak{I}_{3,S}}\int_{1/T<y_1<T} y_1^s E(z_1\,,\,t)E(z_1\,,\,1-t) \times y_1^{1-s}h(s\,,\,t)\,\frac{dy_1}{y_1^2}\,d\mu(z_1)\,ds\,dt\,.$$

We can take $\mathrm{Re}\, t = c_2 = \frac{1}{2}$ because we only needed $\mathrm{Re}\, s$ large. Integrating out y gives (54) equal to

$$2\ln T\int_{\mathrm{Re}\, t=1/2}\int_{z\in\mathfrak{I}_{3,S}} \check{h}(t)E(z_1\,,\,t)\overline{E(z_1\,,\,t)}\,d\mu(z)\,dt$$

where

$$\check{h}(t) = \int_{\mathrm{Re}\, s=c} h(s\,,\,t)\,ds\,.$$

The other two terms behave differently. Moving lines of integration and doing some messy integrals (see [29] for details) gives an additional amount of $K_1 h(\frac{1}{2}\,,\,\frac{1}{2})$ where K_1 is some constant which is independent of all variables in sight.

In [29] we also calculate the residues of $g(s\,,\,t)$, (53) and see that they are zero.

The last Eisenstein series to be considered is the residue of $E(Y\,,\,s\,,\,t)$ as s and t move from the region of convergence to $\mathrm{Re}\, s = \mathrm{Re}\, t = \frac{1}{2}$. Looking at Figure 1 tells us that poles occur at $t = 1$ and at $s = \frac{5}{6}$, $t = \frac{1}{2}$ if we follow a path from $\mathrm{Re}\, t = c_1$ to $\mathrm{Re}\, t = \frac{1}{2}$ (fixing s) and then from $\mathrm{Re}\, s = c_2$ to $\mathrm{Re}\, s = \frac{1}{2}$. When moving past $t = 1$ we pick up a residue $\mathrm{Res}_{t=1} E(Y\,,\,s\,,\,t)$ which we will call $E_R(Y\,,\,s)$. We can then move $E_R(Y\,,\,s)$ and $\mathrm{Re}\, s = \frac{1}{2}$. As we do this we pass through two places where $E_R(Y\,,\,s)$ has poles, namely at $s = 1$ and $s = \frac{2}{3}$. At these places $E_R(Y\,,\,s)$ is equivalent via the functional equations to the residue of $E(Y\,,\,s\,,\,t)$ at $\mathrm{Re}\, s = \frac{5}{6}$, $\mathrm{Re}\, t = \frac{1}{2}$.

We need to consider first the integral

$$\int_{\mathfrak{I}_{T,S}}\int_{\mathrm{Re}\, s=1/2} h(s\,,\,1)E_R(Y\,,\,s)\overline{E_R(Y\,,\,s)}\,ds\,d\mu(z) \tag{55}$$

and the residues of the $E_R(Y, s)$ at $\operatorname{Re} s = 1$ and $\operatorname{Re} s = \frac{2}{3}$. Note that $h(s, 1) = h(s, 0)$ because t and $1-t$ are equivalent under the functional equations and so t and $1-t$ give the same eigenvalue of the pertinent differential operators.

Neglecting poles of $E_R(Y, s)$ for the moment we will compute (55). As usual, replace s by $1-s$, treat the resulting function

$$g_R = E_R(Y, s)E_R(Y, 1-s)h(s, 1)$$

as a meromorphic function of s and move the line of integration to $\operatorname{Re} s$ large. The function $g_R(z, s)$ has three poles along the real axis in s. Because $E_R(z, s)$ has poles at $s = \frac{2}{3}$, $s = 1$, $E_R(z, 1-s)$ has poles at $\frac{1}{3}$ and 0. So we must check for residues at $\frac{1}{3}$, $\frac{2}{3}$, and at 1. This calculation is done in [29] and yields a contribution of $K_2 h(\frac{1}{3}, 1) + K_3 h(\frac{2}{3}, 1)$ where, again K_2 and K_3 are constant.

Now we can proceed to unwind g_R for $\operatorname{Re} s$ large. For $\operatorname{Re} s$ large the Eisenstein series with constant term $y_1^s c_E + \cdots$ is just the one you get by summing $c_E \sum_{\gamma \in \Gamma_{1,\infty}\backslash\Gamma} y_1(\gamma z_1)^s \cdot 1$ (where $c_E = \operatorname{Res}_{t=1} E(z_1, t_1)$) which converges for $\operatorname{Re} s$ large enough. (See Imai and Terras [12].) So we can unwind $E_R(s, z)$ to obtain

$$\int_{\operatorname{Re} s=c} \int_{1/T<y_1<T} \int_{z_1 \in \mathfrak{F}_{3,S}} \int_{1<x_1, t_1<1} y_1^s c_E E_R(Y, 1-s)\, d\mu(Y)\, ds.$$

Again, we are suppressing a calculation which is done in Appendix 2. The above, after integrating out x_1 and t_1, becomes

$$\int_{\operatorname{Re} s=c} \int_{1/T<y_1<T} \int_{z_1 \in \mathfrak{F}_{3,s}} y_1^s c_E \left[y_1^{1-s} c_E + y_1^{s/2} c_1 c\left(\frac{3-3s}{2}\right) E\left(z_1, \frac{2-3s}{2}\right)\right] \times h(s, 1)\, d\mu(z_1) \frac{dy_1}{y_1^2}\, ds.$$

This expression is the sum of two terms, the first being

$$\int_{\operatorname{Re} s=c} \int_{1/T<y_1<T} \int_{z_1 \in \mathfrak{F}_{3,S}} \frac{c_E^2}{y_1} h(s, 1)\, d\mu(z_1)\, dy_1\, ds$$

which yields

$$c_E^2 \operatorname{vol} \mathfrak{F}_{3,S} \cdot 2 \ln T \int_{\operatorname{Re} s=c} h(s, 1)\, ds, \tag{56}$$

and we can move the line of integration to $\operatorname{Re} s = 1/2$ and replace $h(s, 1)$ by $h(s, 0)$ if we want.

We show in [29] that the second term contributes zero to the formula.

Now, this calculation of E_R is sufficient to cover the other two poles also because the functional equations relate the three lines

(1) $\quad s = \frac{1}{3} + i\alpha, \quad t = 1,$

(2) $\quad s = \frac{5}{6} + i\alpha, \quad t + \frac{1}{2} + i\beta, \quad 3\alpha + \beta = 0,$

(3) $\quad s = \frac{5}{6} + i\alpha, \quad t = \frac{1}{2} + i\beta, \quad 3\alpha - \beta = 0.$

Functional equation (1) says

$$E_R(Y, \tfrac{5}{6} + i\alpha, \tfrac{1}{2} - i3\alpha) = c(1 - (\tfrac{1}{2} + i3\alpha))E_R(Y, \tfrac{5}{6} + i\alpha, \tfrac{1}{2} + i3\alpha)$$

and functional equations (2) says

$$E_R(y\,,\,\tfrac{5}{6}+i\alpha\,,\,\tfrac{1}{2}+i3\alpha) = c\left(1-\left(\frac{3(\frac{5}{6}+i\alpha)-(\frac{1}{2}+i3\alpha)}{2}\right)\right)E_R(Y\,,\,\tfrac{1}{3}+\tfrac{1}{2}i(1-4\alpha)\,,\,1)\,.$$

So the sum of both residues is

$$c(0)(1+c(\tfrac{1}{2}-i3\alpha))E_R(z_1\,,\,\tfrac{1}{3}+\tfrac{1}{2}i(1-4\alpha)\,,\,1)\,.$$

But $c(0)=0$, so the only contribution comes from the term already computed in (56).

So, the terms which contribute $O(\ln T)$ to the parabolic terms of the trace formula are (with appropriate constants put in)

$$2\pi i\left[\frac{-1}{2\pi i}\int_{\mathrm{Re}\,t=1/2}\int_{z_1\in\mathfrak{I}_{3,S}}\check{h}(t)E(z_1\,,\,t)\overline{E(z_1\,,\,t)}\,d\mu(z_1)\,dt + c_E^2\operatorname{vol}\mathfrak{I}_{3,S}\left(\frac{-1}{2\pi i}\right)\int_{\mathrm{Re}\,s=1/2}h(s\,,\,0)\,ds\right].$$

The terms which are 0(1) come from the various residues discussed earlier, and they are

$$K_1h(\tfrac{1}{2}\,,\,\tfrac{1}{2})+K_2h(\tfrac{1}{3}\,,\,1)+K_3h(\tfrac{2}{3}\,,\,1) \tag{57}$$

where the constants are independent of everything in sight, namely z, s, t, h, etc. They are numbers. All other terms are $o(1)$ and have been suppressed completely.

6. The rank one cancellation

We cannot claim to have a trace formula for an operator unless we can show that both sides of the formula converge to something finite and independent of any parameters of truncation introduced. The obstacle we must overcome is the coefficient of $\ln T$ in all the "parabolic" terms computed in §§4 and 5.

The reader will by now have noticed the repeated references to the trace formula for $GL(2\,,\,\mathbb{Z})$ acting on $SL(2\,,\,\mathbb{R})/SO(2\,,\,\mathbb{R})$. Bits and pieces of this formula, in fact of the difference between the two sides of the formula, have been appearing among our calculations. Now our task will be to show that all parts of it are present and sum to zero. This calculation will remove most of the badly divergent ($O(\ln T)$) terms in the formula.

By way of review, the $O(\ln T)$ terms appearing thus far are given in formulae (34), (35), (40), (44)–(47), (50), (54), and (56). In this section we will consider all but (44) and (56).

Formula (34) accounts for all of the hyperbolic terms, with a factor of $2\ln T$. The reader is invited to compare this formula with Selberg's original recipe.

Formula (35) accounts for elliptic terms where the element of $GL(2\,,\,\mathbb{Z})$ has distinct nonreal eigenvalues. Again, the formula matches Selberg's with a factor of $2\ln T$. Remember, $\check{h}(t)$ is the Selberg transform (in $SL(2\,,\,\mathbb{R})$) of $\hat{f}(z_1)$.

Formula (40) gives the contribution of the parabolic orbital integrals for which γ has eigenvalues equal to 1, with the truncation parameter $\mathrm{Im}(z)\le S$.

Formula (47) gives the parabolic orbital integrals with eigenvalues equal to -1. Together (40) and (47) give all the parabolic orbital integrals with a factor.

Notice in formulas (40) and (47) that we sum over elements of the form $\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}-1 & a\\ 0 & -1\end{smallmatrix}\right)$ where $a > 0$ only. This is because in $GL(2, \mathbb{Z})$ the elements $\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & -a\\ 0 & 1\end{smallmatrix}\right)$, for example, are conjugate. Notice also that since f is left invariant by $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$ the two formulas are equal. So we get a total contribution of

$$\left(4 + \frac{1}{2}\right) \ln T \sum_{a>0} \int_{u=1/S}^{S} \hat{f}\begin{pmatrix}1 & a/u\\ 0 & 1\end{pmatrix} \frac{du}{u^2}.$$

Now, this amount is more than you would expect from elements conjugate to $\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}-1 & a\\ 0 & -1\end{smallmatrix}\right)$. This is because the formula gives the same amount you would get by starting with a fundamental region for $SL(2, \mathbb{Z})$ rather than $GL(2, \mathbb{Z})$. Thus we have

$$\frac{1}{2} \ln T \sum_{a>0} \int_{u=1/S}^{S} \hat{f}\begin{pmatrix}1 & a/u\\ 0 & 1\end{pmatrix} \frac{du}{u^2} \tag{58}$$

left to explain. The part which contributes to the lower rank trace is

$$4 \ln T \sum_{a>0} \int_{u=1/S}^{S} \hat{f}\begin{pmatrix}1 & a/u\\ 0 & 1\end{pmatrix} \frac{du}{u^2}. \tag{59}$$

Formula (50) subtracts the trace of the operator formed by using the function

$$\sum_{\gamma \in GL(2, \mathbb{Z})} \hat{f}(z^{-1}\gamma w) = \widehat{K}(z, w)$$

as the convolution kernel for the trace formula. Formula (50) subtracts the left-hand side of the $GL(2, \mathbb{Z})$ from all the other terms, which are the right-hand side. This is the key to why cancellation occurs.

Formula (54) calculates the contribution of the Eisenstein series for $GL(2, \mathbb{Z})$ with appropriate factor of $2 \ln T$.

Formulas (45) and (46) together give

$$2 \ln T \cdot 4 \operatorname{vol} \mathfrak{J}_2 \hat{f}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}.$$

Now, since $\hat{f}$ is left and right $SO(2, \mathbb{R})$ invariant, this is the same as

$$2 \ln T \left[2 \operatorname{vol} \mathfrak{J}_2 \hat{f}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} + 2 \operatorname{vol} \mathfrak{J}_2 \hat{f}\begin{pmatrix}+1 & 0\\ 0 & +1\end{pmatrix}\right]. \tag{60}$$

The expression in brackets is the term for the center of $GL(2, \mathbb{Z})$.

Even with these terms we are still missing a few orbital integrals from the rank 1 formula, namely these corresponding to the elements $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}-1 & 1\\ 0 & 1\end{smallmatrix}\right)$. These have simple centralizers. The first one has a centralizer of order 4 and so does the second, so we must add to the $GL(2, \mathbb{Z})$ formula:

$$\begin{aligned} 2 \ln T \left[\frac{1}{4} + \frac{1}{4}\right] \int_{\mathscr{H}} \hat{f}\Bigg(&\begin{pmatrix}1/\sqrt{y} & 0\\ 0 & \sqrt{y}\end{pmatrix}\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix} \\ &\times \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}\Bigg) \frac{dx\,dy}{y^2}. \end{aligned} \tag{61}$$

We can sum them like this because they are all conjugate in $GL(2, \mathbb{Z})$ to $\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$. Truncating y at S and calculating the integral gives

$$2\ln T \cdot \frac{1}{2} \cdot 2\ln S \cdot \frac{1}{2}\int_{u=0}^{\infty} \hat{f}\begin{pmatrix}-1 & u\\ 0 & 1\end{pmatrix} du$$

which is equal by left $SO(2)$ invariance to

$$\frac{1}{2}(\ln T)(\ln S)\int_{u=0}^{\infty} \hat{f}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} du. \tag{62}$$

Of course, adding this in obliges us to subtract it off also. Fortunately, it cancels with (58) as $\ln S$ gets large.

Last we must look for the contribution to the spectrum of the constant function. For this we must subtract off and later add it back in. It will contribute

$$2\ln T(\operatorname{vol}\mathfrak{I}_2)^{-1}\check{h}(0). \tag{63}$$

To summarize, if we add up (34), (35), (40), (47), (58), (50), (54), (45), (46), (62), and (63), we get *zero* because they are, taken together, equal to

$$2\ln T[[\text{right-hand side } GL(2, \mathbb{Z}) \text{ trace formula}] - [\text{left-hand side } GL(2, \mathbb{Z}) \text{ trace formula}]].$$

Now you see why this section is called the rank one cancellation.

7. The rank zero cancellation

Terms left in the parabolic part of the trace formula after §6 come from formulae (44), (56), and (63). Together they give

$$\begin{aligned}&\zeta(2)\ln T\int_{v=0}^{\infty} f\left(\begin{pmatrix}1 & 0 & v\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right) v\,dv\\ &\quad - 2c_E^2 \operatorname{vol}\mathfrak{I}_3 \cdot \ln T\left(\frac{1}{2\pi i}\right)\int_{\operatorname{Re}s=c} h(s, 1)\,ds\\ &\quad + 2\ln T\cdot \check{h}(0)\cdot\frac{1}{\operatorname{vol}\mathcal{I}_2}.\end{aligned} \tag{64}$$

Using the functional equations to replace $h(s, 1)$ by $h(s, 0)$ introduces a factor of $6/\pi$. Thus the second term of the summand is

$$-2c_E^2 \operatorname{vol}\mathfrak{I}_3\left(\frac{6}{\pi}\right)\ln T\left(\frac{1}{2\pi i}\right)\int_{\operatorname{Re}s=c} h(s, 0)\,ds. \tag{65}$$

The integral $(1/2\pi i)\int_{\operatorname{Re}s=1/2} h(s, 0)\,ds$ can be expanded as

$$\frac{1}{2\pi i}\int_{\operatorname{Re}s=c}\left[\int_{Y\in SL(3,\mathbb{R})/SO(3,\mathbb{R})} f(Y)y(Y)^{6s}u(Y)^0\,dY\right] ds.$$

Writing the part in brackets in geodesic polar coordinates for $SL(3, \mathbb{R})$ gives

(66)

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\operatorname{Re}s=c}\int_{K\in SO(3)}\frac{\pi^2}{3\cdot 8}\int_{y,u>0} f\left(K\begin{pmatrix}yu^{1/2} & 0 & 0\\ 0 & yu^{-1/2} & 0\\ 0 & 0 & y^{-2}\end{pmatrix}\right) y(KY)^{6s}\\ &\quad\times(yu^{1/2} - yu^{-1/2})(yu^{1/2} - y^{-2})(y^{-2} - yu^{-1/2})\frac{du}{y}\,dK\,ds\,\frac{du}{u}.\end{aligned}$$

For a computation of this Jacobian, see Terras [24]. The mysterious factor of 6 in the exponent comes from setting the y of formula (3) equal to y instead of $y^{1/6}$. Now, viewing the integrals in s and y, we see that this is the Helgason transform in y followed by its inverse at $y = 1$. The integral in (65) then becomes

$$\int_{K\in\mathscr{SO}(3)}\int_{u>0} f\left(\begin{pmatrix} u^{1/2} & 0 & 0 \\ 0 & u^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \times (u^{1/2}-u^{-1/2})(u^{1/2}-1)(1-u^{-1/2})\,dK\,\frac{du}{u}. \tag{67}$$

Now, f is $\mathscr{SO}(3)$-bi-invariant, so one can eliminate or replace K with any special value desired. We would like to rotate the point

$$\begin{pmatrix} u^{1/2} & 0 \\ 0 & u^{-1/2} \end{pmatrix} \circ i = ui$$

in the Poincaré upper half-plane so that its imaginary part is one. In other words, we want

$$\operatorname{Im}\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \circ ui\right) = 1.$$

Computing the point gives

$$\begin{aligned}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \circ ui &= \frac{(1-u^2)\sin\theta\cos\theta}{\cos^2\theta + u^2\sin^2\theta} + i\left(\frac{u}{\cos^2\theta + u\sin^2\theta}\right) \\ &= \frac{(1-u^2)(\frac{1}{2})\sin 2\theta}{\frac{1}{2}(u^2+1+(1-u^2)\cos 2\theta)} + i\left(\frac{u}{\frac{1}{2}(u^2+1+(1-u^2)\cos 2\theta)}\right).\end{aligned}$$

If the imaginary part of this number is one then the real part is given by

$$\operatorname{Re}(k_\theta \cdot ui) = \frac{1}{2u}(1-u^2)\sin 2\theta.$$

Furthermore, $\cos 2\theta = (u-1)/(u+1)$. With these substitutions,

$$\begin{aligned} k_\theta \circ iu &= i + \left(\frac{(1-u^2)\frac{1}{2}\sin(2\theta)}{\frac{1}{2}(\cos 2\theta(1-u^2)+1+u^2)}\right) \\ &= i + u^{-1/2}(1-u).\end{aligned}$$

Setting $v = u^{1/2} - u^{-1/2}$, $dv = \frac{1}{2}(u^{-3/2}+u^{-1/2})\,du$. So

$$v\,dv = \frac{1}{2}(u^{1/2}-u^{-1/2})(u^{-3/2}+u^{-1/2})\,du = \frac{1}{2}(u-u^{-1})\,\frac{du}{u}.$$

Checking that, in (67),

$$(u^{1/2}+u^{-1/2})(u^{1/2}+1)(1-u^{-1/2}) = (u-u^{-1}) + (u^{-1/2}-u^{1/2})$$

we see that (65) equals

(68)

$$-2(c_E^2)\operatorname{vol}\mathfrak{I}_3 \ln T\left(\frac{6}{\pi}\right)\left(\frac{\pi^2}{3\cdot 8}\right)$$

$$\times\left[\int_u\int_{K\in SO(3)} f\left(K\begin{pmatrix} u^{1/2} & 0 & 0\\ 0 & u^{-1/2} & 0\\ 0 & 0 & 1\end{pmatrix}\right)(u^{-1/2}-u^{1/2})\frac{du}{u}\,dk\right.$$

$$\left.+2\int_{v=0}^{\infty} f\left(\begin{pmatrix} 1 & 0 & v\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right) v\,dv\right].$$

Now, $c_E+\pi/3=(\operatorname{vol}\mathfrak{I}_3)^{-1}$, so simplifying the constants gives

$$-2c_E^2\operatorname{vol}\mathfrak{I}_3\left(\frac{6}{\pi}\right)\left(\frac{\pi^2}{3\cdot 8}\right)\cdot 2 = -2\cdot\left(\frac{\pi}{2\cdot 3}\right)^2\left(\frac{3\cdot 2}{\pi}\right)\left(\frac{6}{\pi}\right)\left(\frac{\pi^2}{3\cdot 8}\right)\cdot 2$$

$$=\frac{\pi^2}{6}=\zeta(2).$$

Therefore the second summand in (68) cancels out the term form (44). We are left in (68) with the integral

$$\text{(69)}\qquad \frac{\pi^2}{12}\ln T\int_{u>0} f\left(\begin{pmatrix} u^{1/2} & 0 & 0\\ 0 & u^{-1/2} & 0\\ 0 & 0 & 1\end{pmatrix}\right)(u^{-1/2}-u^{1/2})\frac{du}{u}$$

which we hope cancels with formula (63) which appears as the third summand in (64).

But we notice that the factor $(u^{1/2}-u^{-1/2})/u$ in (69) is just the Jacobian for geodesic polar coordinates in the Poincaré upper half-plane. So we have that

$$\text{(70)}\qquad \int_{u>0} f\left(\begin{pmatrix} u^{1/2} & 0 & 0\\ 0 & u^{-1/2} & 0\\ 0 & 0 & 1\end{pmatrix}\right)(u^{1/2}-u^{-1/2})\frac{du}{u}.$$

After writing f in terms of its Selberg transform at u, $y=$ yields

$$=\frac{-1}{4\pi^2}\int_{\operatorname{Re}t=1/2}\int_{\operatorname{Re}s=1/2}\int_{u>0} h(s,t)(u^{1/2}-u^{-1/2})u^t\frac{du}{u}\,ds\,dt.$$

Integrating s and throwing in a gratuitous rotation gives

$$=\frac{+1}{2\pi i}\int_{\operatorname{Re}t=1/2}\check{h}(t)\left[\int_{u>0}\int_{\theta=0}^{\pi}(u^{1/2}-u^{-1/2})u^t\frac{du}{u}\frac{d\theta}{\pi}\right]dt.$$

Writing u as $\operatorname{Im}(z)$ and changing back to rectangular coordinates gives

$$\left(\frac{2}{\pi}\right)\frac{1}{2\pi i}\int_{\operatorname{Re}t=1/2}\int_z \check{h}(t)\operatorname{Im}(z)^t\,dt\,d\mu(z)=\frac{2}{\pi}\int_z\tilde{f}(z)\,d\mu(z)=\frac{2}{\pi}\cdot 2\check{h}(0).$$

Thus, (69) becomes

$$\ln T\frac{\pi^2}{12}\frac{4}{\pi}\check{h}(0)=2\ln T\left(\frac{\pi}{6}\right)\check{h}(0)=2\ln T\left(\frac{1}{\operatorname{vol}\mathfrak{I}_2}\right)\check{h}(0)$$

which is what must be cancelled from (63). In short, (64) is zero.

This concludes our discussion of the parabolic term. The reader will notice that our techniques give more cancellation than expected. Presumably one could apply these techniques to the classical formula in order to show certain of the terms in it are actually zero.

8. Statement of the theorem

Let $G = SL(3, \mathbb{R})$, $K = SO(3, \mathbb{R})$, and $\Gamma = SL(3, \mathbb{Z})$. Let f be a left K-invariant function on G/K and let f infinitely differentiable and compactly supported. From the convolution kernel

$$K(Z, Y) = \sum_{\gamma \in \Gamma} f(Z^{-1}\gamma Y)$$

and denote by $L_{\tilde{k}}$ the integral operator on $L^2(\Gamma\backslash G/K)$ built from kernel via formula (16). Then this operator is trace class and the trace of $L_{\tilde{k}}$ on the discrete joint spectrum of the invariant differentiable operators on this space is given by the following formula:

$$\begin{aligned}
\mathrm{Tr}(L_k) &= \mathrm{vol}(\mathfrak{I}) f(I) \\
&+ \sum_{\substack{\text{totally real} \\ \text{cubic number fields} \\ \text{with fundamental} \\ \text{units } \alpha_1, \beta_1}} \sum_{j, k \neq (0,0)} \mathrm{cl}(\mathbb{Z}[\alpha_1]) \operatorname{Re} g(\mathbb{Z}[\alpha_1]) \left| \frac{(\alpha_1^j \beta_1^k)^2 (\alpha_2^j \beta_2^k)}{W(\alpha_1^j \beta_1^k)} \right| \hat{\hat{f}}(A^j B^k) \\
&+ \sum_{\substack{\mathbb{Z}[r_0 e^{i\theta_0}] \\ \text{distinct} \\ \text{mixed cubic} \\ \text{numbers fields}}} \sum_{j>0} \frac{\mathrm{cl}(r_0, \theta_0) |\ln r_0|}{|1 - 2r_0^{-3j} \cos j\theta + r_0^{-6j}|} \\
&\times \int_{\operatorname{Re} t = 1/2} \int_{\operatorname{Re} s = 1/2} h(s, t) \frac{r_0^{j(1-s)} e^{-2j\theta_0 \operatorname{Im} t}}{1 + e^{-2\pi t}} \, ds \, dt \\
&+ (K_1) h\left(\frac{1}{2}, \frac{1}{2}\right) + K_2 h\left(\frac{1}{3}, 1\right) + K_3 h\left(\frac{2}{3}, 1\right)
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{vol}\, \mathfrak{I} &= \text{volume of a fundamental region for } \Gamma \text{ on } G/K, \\
A &= \begin{pmatrix} \alpha_1 & 0 & \phi \\ 0 & \alpha_2 & 0 \\ 0 & 0 & 1/\alpha_1\alpha_2 \end{pmatrix}, \quad B = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & 1/\beta_1\beta_2 \end{pmatrix}, \\
W(\alpha_1, \beta_1) &= (\alpha_1\beta_1 - \alpha_2\beta_2)(\alpha_2\beta_2 - \alpha_3\beta_3)(\alpha_1\beta_1 - \alpha_3\beta_3), \\
\alpha_3 &= 1/\alpha_1\alpha_2, \quad \beta_3 = 1/\beta_1\beta_2, \\
\mathrm{cl}(R) &= \text{ class number of ring } R, \\
\operatorname{Re} g(R) &= \text{regulator of } R, \\
f(E) &= \int_N f(EN) \, dN, \\
N &= \begin{pmatrix} 1 & v & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad dN = dv\, dx\, dt,
\end{aligned}$$

$$r_0 e^{i\theta_0} = \text{fundamental unit of a mixed cubic number field},$$
$$h(s, t) = \text{the Selberg transform of } f \text{ at the eigenvalues, corresponding to } s \text{ and } t,$$
$$j\theta_0 \text{ is taken mod } \pi.$$

Proof. These are the terms left after cancellation, namely, (19), (21), (28), and (57).

Appendix 1

We must calculate the number of conjugacy classes associated to an element in P_1.

In order to justify formula (34) we must prove (33) here. We need to count the number of $SL(3, \mathbb{Z})$ conjugates of an element with eigenvalues $\{\varepsilon, \tilde{\varepsilon}, \eta(\varepsilon)\}$ where ε is a unit other than ± 1 is a quadratic number field, $\tilde{\varepsilon}$ is its conjugate and $\eta(\varepsilon) = \pm 1$ is the norm $\varepsilon\tilde{\varepsilon}$. We already showed in [27] that such an element is conjugate to something of the form

$$A = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{M} & a \\ & & b \\ \hline 0 & 0 & \pm 1 \end{array}\right), \qquad M \in GL(2, \mathbb{Z}).$$

Since P_1 is its own normalizer we need only consider further conjugacy by elements of P_1. Conjugating by an element of the form

$$\left(\begin{array}{cc|c} \multicolumn{2}{c|}{B} & 0 \\ & & 0 \\ \hline 0 & 0 & \pm 1 \end{array}\right)$$

allows us to choose a particular M as representative of the $GL(2, \mathbb{Z})$ conjugacy class associated to $\{\varepsilon, \tilde{\varepsilon}\}$ in $GL(2, \mathbb{R})$. Therefore the total number of conjugacy classes is $\mathrm{cl}(\varepsilon, \tilde{\varepsilon}) \cdot n$ where n is the number of inequivalent matrices with a particular choice of M in the upper left-hand corner. In order to count the number of inequivalent γ of the form

$$\gamma = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{M} & a \\ & & b \\ \hline 0 & 0 & 1 \end{array}\right)$$

with fixed M, one must conjugate by

$$\gamma' = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{Q^k} & c \\ & & d \\ \hline 0 & 0 & 1 \end{array}\right)$$

where Q generated the centralizer of M in $SL(2, \mathbb{Z})$. Explicitly, this conjugation gives

$$\gamma'\gamma(\gamma')^{-1} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{Q^k} & c \\ & & d \\ \hline 0 & 0 & 1 \end{array}\right) \left(\begin{array}{cc|c} \multicolumn{2}{c|}{M} & a \\ & & b \\ \hline 0 & 0 & 1 \end{array}\right) \left(\begin{array}{cc|c} Q^{-k} & -Q^{-k} & \binom{c}{d} \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{M} & * \\ \hline 0 & 0 & 1 \end{array}\right)$$

where

$$* = -M\begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} + Q^k \begin{pmatrix} a \\ b \end{pmatrix}. \tag{71}$$

To count the inequivalent candidates we will start with the case $k=0$. Then we can reduce $\binom{a}{b}$ by adding on something of the form $(I-M)\binom{c}{d}$. We can compute $\det(I-M)$, which gives $(2-\operatorname{Tr} M)$ if $\det M=1$ and $\operatorname{Tr} M$ if $\det M=-1$. The case where $\operatorname{Tr} M=0$ and $\det M=-1$ is covered elsewhere, and $\operatorname{Tr} M \neq 2$ for these terms. Thus, $I-M$ is invertible. Now we can put $I-M$ into Smith normal form which gives

$$(I-M)=\begin{pmatrix} d_1 & e \\ 0 & d_2 \end{pmatrix} N, \qquad d_1 d_2=\det(I-M),\ \det N=1,\ 0 \leq e < d_1 .$$

We can now choose $\binom{c}{d}$ so that

$$N\begin{pmatrix} c \\ d \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then

$$(I-M)\begin{pmatrix} c \\ d \end{pmatrix}=\begin{pmatrix} e \\ d_2 \end{pmatrix}$$

and thus

$$\begin{pmatrix} a \\ b \end{pmatrix} \cong \begin{pmatrix} a' \\ b \bmod d_2 \end{pmatrix}.$$

Similarly, by choosing $\binom{c}{d}$ so that

$$N\begin{pmatrix} c \\ d \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

we have

$$(I-M)\begin{pmatrix} c \\ d \end{pmatrix}=\begin{pmatrix} d_1 \\ 0 \end{pmatrix}$$

so that

$$\begin{pmatrix} a \\ b \end{pmatrix} \cong \begin{pmatrix} a^1 \bmod d_1 \\ b \bmod d_2 \end{pmatrix}.$$

Thus we have at most $2-\operatorname{Tr} M$ conjugacy classes corresponding to M.

Now we must account for the effect of Q^k.

Recall that conjugation by an element of the form

$$\left(\begin{array}{cc|c} \multicolumn{2}{c|}{Q^k} & c \\ & & d \\ \hline 0 & 0 & 1 \end{array}\right)$$

leads to the expression $(I-M)\binom{c}{d}+Q^k\binom{a}{b}$ in the upper right-hand corner of our new matrix. If two of these agree, we have

$$(I-M)\begin{pmatrix} c_1 \\ d_1 \end{pmatrix}+Q^{k_1}\begin{pmatrix} a \\ b \end{pmatrix}=(I-M)\begin{pmatrix} c_2 \\ d_2 \end{pmatrix}+Q^{k_2}\begin{pmatrix} a \\ b \end{pmatrix}$$

or equivalently,

$$(I-M)\begin{pmatrix} c_1-c_2 \\ d_1-d_2 \end{pmatrix}+(Q^{k_1}-Q^{k_2})\begin{pmatrix} a \\ b \end{pmatrix}=0 .$$

Remembering that Q generates the centralizer of M, we will assume $k_2>k_1$ and write (71) as

$$Q^{k_1}\left[(I-M)\begin{pmatrix} c \\ d \end{pmatrix}+(I-Q^k)\begin{pmatrix} a \\ b \end{pmatrix}\right]=0$$

where

$$\begin{pmatrix} c \\ d \end{pmatrix} = Q^{-k_1} \begin{pmatrix} c_1 - c_2 \\ d_1 - d_2 \end{pmatrix}$$

and $k = k_2 - k_1$. This yields

$$\begin{pmatrix} -c \\ -d \end{pmatrix} = (I - M)^{-1}(I - Q^k)\begin{pmatrix} a \\ b \end{pmatrix}$$

which is a situation we can achieve exactly when $(I - M^{-1})(I - Q^k)\binom{a}{b}$ is an integer vector. Referring to the results in (26) we see that k is exactly the power for which the generator of the centralizer for

$$\begin{pmatrix} M & & a \\ & & b \\ 0 & 0 & 1 \end{pmatrix}$$

looks like

$$\left(\begin{array}{cc|c} Q^k & & c \\ & & d \\ \hline 0 & 0 & 1 \end{array}\right).$$

Thus the number of conjugacy classes with M in the upper left-hand corner is

$$\sum_{0 \le a < d_1} \sum_{0 \le b < d_2} \frac{1}{K_{a,b,M}}, \qquad d_1 d_2 = |2 - \operatorname{Tr} M|,$$

where $k_{a,b}$ is the least k such that

$$(I - M^{-1})(I - Q^k)\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^2.$$

Therefore the expression in (34)

$$\begin{aligned} \xi(\lambda, \tilde{\lambda}, \eta(\lambda)) &= \ln \varepsilon \sum_{i=0}^{\mathrm{cl}(\lambda, \tilde{\lambda}, \eta(\lambda))} k_i \\ &= \ln \varepsilon \sum_{\{M\}} \sum_{0 \le a < d_1} \sum_{0 \le b < d_2} \frac{1}{k_{a,b,m}} \cdot k_{a,b,m} \\ &= \ln \varepsilon \, \mathrm{cl}(\lambda, \tilde{\lambda}) |2 - \operatorname{Tr} M| \end{aligned}$$

where $\mathrm{cl}(\lambda, \tilde{\lambda})$ is the narrow class number of the order $\mathbb{Z}[\lambda]$. This proves the claim in (33).

APPENDIX 2. REGIONS OF INTEGRATION FOR THE PARABOLIC TERM

This appendix is inserted to justify a computational trick that was used in [27, 28, 29, and 31]. Sections 4 and 5 are based on the use of this trick, which is easy to justify only when all parts of the trace formula are present. For the parabolic orbital integrals and the Eisenstein series both, we took the liberty of replacing an integral over the region $\{\bigcup_{\gamma \in P_1 \backslash \Gamma} \gamma^{-1} \mathfrak{I}_{T_1 s} = R_1\}$ with the region

$$\left\{ \begin{array}{c} 0 < x_2 < 1 \\ 0 < x_2 < 1 \\ z \in \mathfrak{I}_{3,S} \\ 1/T < y < T \end{array} \right\} = R_2.$$

In the situation where the difference between the integrals over R_1 and R_2 was estimated explicitly, namely [28], the difference was $o(T)$ and thus could be ignored for our purposes. In other situations, namely [27], [29], and [31], the author's fortitude flagged noticeably when the moment came to do this estimate, resulting in a broad hint that any reader with his salt could go check it himself. Our apologies to any reader who actually attempted to do this. It is much easier to justify this trick in the context of the entire parabolic term.

A quick perusal of §§4 and 5 shows that the above trick of replacing R_1 by R_2 was done in every single integral in what is called the parabolic term. What must be shown to justify it is that

$$\int_{R_1} \text{(paraoblic term)} - \int_{R_2} \text{(parabolic term)} = \int_{R_1 - R_2} \text{(parabolic term)} = o_T(1).$$

Now, the region $R_1 - R_2$ is just given by $R_3 = \{\gamma^{-1}Y : Y \in \mathfrak{J}_{T,S},\ \gamma \in P_1\backslash\Gamma,\ \gamma^{-1}Y < 1/T\}$ and $R_3 = \bigcup S_n$ where $S_n = \{\gamma^{-1}Y : Y \in \mathfrak{J}_{T,S},\ \gamma \in P_1\backslash\Gamma,\ 1/(n+1)T < \gamma^{-1}Y < 1/nT\}$. If the integral $\int_{R_3}$ (parabolic term) is not $o_T(1)$ then the partial sum $\sum_{n=1}^{N} \int_{S_n}$ (parabolic term) must not go to zero in T. We can instead look at

$$\sum_{n=1}^{N} \int_{S_n} |\text{(parabolic terms)}| \tag{72}$$

and it suffices to show that this goes to zero. But (72) is first equal to

$$\int_{\bigcup_{1\le n\le N} S_n} |\text{(parabolic term)}| \le \int_{R_4} |\text{(parabolic term)}|$$

where $R_4 = \{Y : 0 \le x_1,\ x_2 \le 1,\ z \in \mathfrak{J}_{3_1 s},\ 1/(n+1)T \le y \le 1/T\}$. That is to say, $\bigcup_{n=1}^{N} S_n$ is contained in the portion of the cylinder containing R_2 where $1/(n+1)T \le y \le 1/T$.

However, a quick review of §§6 and 7 easily convinces us that if we replace the lower endpoint of the y-integral (formerly $1/T$) with some other parameter (in this case, $1/(n+1)T$), the value of the trace formula remains the same. All that will change is that $2\ln T$ will be replaced by $(n+2)\ln T$ as the truncation coefficient in front of the various expressions. Therefore we can conclude that $\int_{R_4}$ |(parabolic term)| is indeed $o_T(1)$ and this, in turn, justifies the trick of replacing R_1 by R_2 in §§4 and 5.

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